Université de Cergy-Pontoise **UFR Sciences et Techniques**

Etude mathématique des propriétés de transport des opérateurs de Schrödinger aléatoires avec structure quasi-cristalline

THÈSE

présentée et soutenue publiquement le 25 Juin 2012

pour l'obtention du grade de

Docteur de l'Université de Cergy-Pontoise

Spécialité Mathématiques

par

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Laboratoire AGM — UMR CNRS 8088

Remerciements

Je tiens tout d'abord à remercier François Germinet, mon directeur de thèse, pour m'avoir encadré et soutenu tout au long de ces années. Ses qualités scientifiques et humaines m'ont profondément marqué et je suis particulièrement admirative devant sa capacité à gérer ses activités au sein de l'université en étant toujours disponible et attentionné. Merci, pour son enthousiasme à toute preuve, pour m'encourager face aux problèmes et se réjouir de mes réussites.

Je tiens également à exprimer toute ma reconnaissance aux rapporteurs de cette thèse, Frédéric Klopp et Peter Hislop, pour leur dévouement et enthousiasme face aux jeunes membres de la communauté du Schrödinger aléatoire. Les cours annuels de Frédéric, ainsi que le programme organisé avec Peter "Spectral and Dynamical Properties of Quantum Hamiltonians" au C.I.B. de l'EPFL– Lausanne, ont été fondamentaux pour mon intérêt pour les modèles quantiques aléatoires.

Je remercie très chaleureusement Laurent Bruneau, Alain Joye et Abel Klein d'avoir accepté de faire partie du jury. C'est un honneur de me voir entourée de chercheurs si brillants, mais c'est surtout leur aspect humain qui me produit une profonde joie de partager ce moment avec eux.

I am grateful to Peter Müller for his hospitality at LMU Munich. I sincerely appreciate his attention and patience towards me during both my stay in Munich and the writing of this thesis. His pedagogical skills and physical insight were of great help.

I wish to thank Ivan Veselić at TU Chemnitz for his hospitality during my visits and for the remarks that helped improve this manuscript. I enjoyed and learned very much from our scientific collaboration. Similarly, I would like to thank Daniel Lenz at Friedrich Schiller Universität Jena, for his hospitality and for stimulating conversations.

Toute ma gratitude va au Laboratoire AGM de l'Université de Cergy-Pontoise, pour m'avoir accueilli pendant le développement de mes recherches. Je remercie en particulier Vladimir Georgescu, pour nourrir un environnement harmonieux et son soutien constant aux thésards, ainsi que Mathieu Lewin et Thierry Jecko, pour leurs conseils et leur disponibilité. Je voudrais remercier spécialement Andrei Iftimovici pour son attention constante et ses encouragements. Merci aux thésards, quelle troupe des gens sympas ! Lysou, Anneso, David, Nico, Baptiste, Séb et les anciens, Jérôme, Christophe et le petit soldat de terrecuite, Nico Dom et ses histoires Bukowskiennes. Un grand merci à Amal, ma petite soeur de thèse et chère complice, pour sa générosité, son soutien et pour avoir partagé les joyeux et les pas-si-joyeux moments.

My sincere thanks go to Francisco Hoecker-Escuti for his advice and long conversations we held during these years. I always enjoyed our dialogues, but mostly, his monologues. Gracias a Jimena y Eduardo por la jornadas en el Institut Henri Poincaré. Gracias a Lili y Arturo, a André, a Sergito Patricio, Cristobal y Alou, mi coach, por acompaãrme en el camino y por su cariño. Gracias a mis amigos en Chile y en Francia. Todo hubiese sido tan vacío sin ustedes.

I cannot thank enough Thanos for his love, tireless support and patience towards my chaos. For making life sweeter. I hope we will spend long years smiling and jumping into puddles together.

A Ximena, mi madre, gracias por apoyarme hasta el infinito y más allá. Gracias por creer en mis ideas y transmitirme esa curiosidad sin límites. Por ser la más entusiasta compañera de

aventuras. A Ximena, mi hermana, gracias por todas las enseñanzas, por incentivarme siempre a hacer lo que me gusta, ser apasionada en la vida y darme el ejemplo. Ustedes son mi gran inspiración y les debo lo que soy. A Elvira y Mario, que siempre han estado con nosotras, gracias por el apoyo incondicional. Los llevo en mi corazón.

I wish to acknowledge Ximena Molina, in particular, for financing my formal education and my Master studies at the Pierre and Marie Curie University. These are times when the situation of education in Chile is critical and most students obtain their degrees together with huge debts for them and their families. Today I am fortunate to say this is not my case. This is because of my mother's many sacrifices in order to afford my education, for which I am forever grateful.

This thesis was financed by a CONICYT Doctoral grant from the Chilean government. Partial financing by ANR BLAN 0261 is gratefully acknowledged. My research visit to LMU Munich was possible thanks to Cergy-Pontoise University's Erasmus Program.

A Ximena, mamá y a Ximena, mi hermana.
A Elvira y Mario.

"This day before dawn I ascended a hill and look'd at the crowded heaven,
And I said to my spirit When we become the enfolders of those orbs, and the
pleasure and knowledge of every thing in them,
shall we be fill'd and satisfied then?
And my spirit said No, we but level that lift to pass and continue beyond."
Walt Whitman, Song of Myself 46.

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Résumé

Cette thèse est consacrée à l'étude du transport électronique dans des modèles désordonnés non ergodiques, dans le cadre de la théorie des opérateurs de Schrödinger aléatoires.

Pour commencer, nous reformulons l'outil principal pour notre étude, l'analyse multi-échelles, dans le cadre non ergodique. Nous établissons les conditions d'homogénéité que l'opérateur doit vérifier pour appliquer cette méthode.

Ensuite, nous étudions les propriétés spectrales des opérateurs de Delone–Anderson non ergodiques. Ces systèmes modélisent l'énergie d'une particule en interaction avec un milieu dont la structure atomique est quasi-cristalline et la nature des impuretés est désordonnée. Dans le cas où les mesures de probabilité associées au potentiel de simple site sont régulières, en dimension 2 et sous l'effet d'un champ magnétique, nous établissons une transition métal-isolant et l'existence d'une énergie de mobilité qui sépare les régions de localisation et de délocalisation dynamiques. Pour des mesures de simple site régulières et celle de Bernoulli, nous démontrons la localisation dynamique en bas du spectre. De plus, nous obtenons une borne inférieure quantitative sur la taille de la région de localisation dynamique en termes des paramètres géométriques de la structure de Delone de base.

Nous concluons ce travail avec l'étude de la densité d'états intégrée pour des modèles de Delone-Anderson, en combinaison avec des outils de la théorie des systèmes dynamiques associés aux quasi-cristaux. Sous certaines conditions sur la géométrie de l'ensemble de Delone sous-jacent, nous montrons l'existence de la densité d'états intégrée. De plus, dans le cas d'une perturbation de Delone-Anderson du Laplacien libre, nous démontrons qu'elle a un comportement asymptotique de Lifshitz en bas du spectre.

Mots-clés : Opérateurs de Schrödinger aléatoires, localisation d'Anderson, transition métal-isolant, localisation dynamique, hamiltonian de Landau, ensembles de Delone, opérateurs de Delone, systèmes dynamiques de Delone, densité d'états intégrée.

Abstract

This thesis is devoted to the study of electronic transport in non ergodic disordered models, in the framework of random Schrödinger operators.

We start by reformulating the main tool in our study, the multiscale analysis, in the non ergodic setting. We establish suitable homogeneity conditions on the operator, in order to apply this method.

Next, we study the spectral properties of non ergodic Delone-Anderson operators. These models represent a particle interacting with a medium whose atomic structure is quasi-crystalline and the nature of its impurities is disordered. In the case where the probability measures associated to the single-site potential are regular, in dimension 2 and under the effect of a magnetic field, we establish a metal-insulator transition and the existence of a mobility edge that separates the localization and delocalization regions. In arbitrary dimension, for regular and for Bernoulli single-site measures, we show dynamical localization at the bottom of the spectrum. Moreover, we obtain a quantitative description of the localization region in terms of the geometric parameters of the underlying Delone set.

We conclude this essay by studying the integrated density of states for Delone-Anderson models, using tools from the theory of dynamical systems associated to quasicrystals. Under certain conditions on the geometry of the underlying Delone set, we show the existence of the integrated density of states. Furthermore, in the case of a Delone-Anderson perturbation of the free Laplacian, we show it exhibits Lifshitz tails at the bottom of the spectrum.

Keywords: Random Schrödinger operators, Anderson localization, metal-insulator transition, dynamical localization, Landau Hamiltonians, Delone sets, Delone operators, Delone dynamical systems, integrated density of states.

Chapitre 1

Introduction et présentation des résultats

1.1 Introduction générale

Ce travail est consacré à l'étude du transport électronique dans des modèles désordonnés non ergodiques issus de la théorie des opérateurs de Schrödinger aléatoires (OSA). On s'intéresse à la dynamique d'une particule dans un espace euclidien de dimension d en regardant l'évolution temporelle de sa fonction d'onde associée ψ appartenant à $\mathrm{L}^2(\mathbb{R}^d)$, avec $\|\psi\|_{\mathrm{L}^2(\mathbb{R}^d)} = 1$. La dynamique est gouvernée par l'équation de Schrödinger

$$i\partial_t\psi(t,x) = H_\omega\psi(t,x), \quad (t,x) \in \mathbb{R}\times\mathbb{R}^d, \omega\in\Omega \tag{1.1.1}$$

$$\psi(t,x) = e^{-itH_\omega}\psi(0,x), \tag{1.1.2}$$

où $\psi(0,x)$ est une condition initiale et l'opérateur $H_\omega = H_0 + V_\omega$, représentant l'énergie de la particule, est associé à un espace de probabilité Ω. Un élément $\omega\in\Omega$ désigne une réalisation possible. La famille $\{H_\omega : \omega\in\Omega\}$ regroupe tous les opérateurs H_ω possibles associés aux réalisations du potentiel V_ω qui représente le milieu désordonné. Ce type d'opérateur fut introduit par P.W. Anderson [And58] qui considéra un modèle dit de liaisons fortes, portant son nom, où les électrons sautent d'un ion à un autre, et interagissent avec un potentiel aléatoire externe qui représente la nature désordonnée du milieu. Anderson soutint l'idée qu'au dessus d'un seuil du nombre d'impuretés présentes dans le cristal, les électrons sont piégés dans des régions bornées, avec pour conséquence la suppression totale du transport électronique. Dans ce cas on parle des états localisés, par opposition aux états dits étendus, propres aux milieux parfaitement périodiques [B28, AK98, RSIV]. La découverte de la *localisation d'Anderson* a ouvert la voie à l'étude des opérateurs de Schrödinger aléatoires. Elle a valu à P.W. Anderson le Prix Nobel de physique en 1977.

Dès ses débuts, l'ergodicité joua un rôle fondamental dans la théorie des OSA, permettant de lier les propriétés de réalisations particulières de H_ω avec des propriétés globales, partagés par (presque) tout élément de la famille $\{H_\omega\}_{\omega\in\Omega}$. Soit T un sous-ensemble additif de $\mathbb{R}^d$ (par exemple, $\mathbb{Z}^d$ ou bien $\mathbb{R}^d$). On dit qu'une famille (muni d'une structure de groupe) $\{\tau_x\}_{x\in T}$

de transformations sur Ω qui préservent la mesure (de probabilité) est *ergodique* si tout sous-ensemble de Ω invariant sous ces transformations a ou bien probabilité 1 ou bien 0. Par ailleurs, on dit que la famille d'opérateurs H_ω est ergodique si elle vérifie une *condition de covariance* par rapport à l'action d'une famille d'opérateurs unitaires (translations) U_γ associée à un groupe ergodique de translations τ_γ agissant sur l'espace de probabilité Ω, à savoir, pour tout $\omega \in \Omega$

$$H_{\tau_x(\omega)} = U_x H_\omega U_x^*, \quad \forall x \in T. \tag{1.1.3}$$

Ceci a des conséquences très importantes sur le spectre $\sigma(H_\omega)$ de H_ω. Cela assure l'existence d'un ensemble $\Sigma \subset \mathbb{R}$ tel que [P80]

$$\sigma(H_\omega) = \Sigma, \quad \text{pour } \textit{presque tout } \omega \in \Omega. \tag{1.1.4}$$

De plus, il existe des ensembles $\Sigma_{pp}, \Sigma_{ac}, \Sigma_{sc} \subset \mathbb{R}$ qui correspondent, respectivement, aux parties purement ponctuelle, singulière continue et absolument continue du spectre de H_ω [KM82a], i.e.

$$\sigma_\bullet(H_\omega) = \Sigma_\bullet, \quad \text{où } \bullet = pp, ac, sc \quad \text{pour } \textit{presque tout } \omega \in \Omega. \tag{1.1.5}$$

En particulier, pour le modèle d'Anderson $H_\omega = \Delta + V_\omega$, le potentiel aléatoire est donné par

$$V_\omega = \sum_{j\in\Gamma} \omega_j u(x-j), \quad \omega_j \in [a,b], \tag{1.1.6}$$

où u est une fonction à support compact dans $\mathbb{R}^d$ ou $\mathbb{Z}^d$ et $\Gamma \subset \mathbb{R}^d$ est un réseau, l'ergodicité est une conséquence directe du fait que les variables aléatoires ω_j sont indépendantes, identiquement distribuées et du fait que Γ est invariant par translation. Pour le modèle d'Anderson, la famille ergodique $\{\tau_x\}$ correspond aux translations $\tau_x(\omega) = (\omega_{j-x})_{j\in\Gamma}$ pour $\omega = (\omega_j)_{j\in\Gamma}$ et $T = \Gamma$. Dans ce cas, le spectre $\sigma(H_\omega)$ peut être caractérisé par le spectre d'une sous-famille de $\{H_\omega\}_{\omega\in\Omega}$, à savoir, les réalisations périodiques de V_ω [KM82b, S],

$$\sigma(H_\omega) = \bigcup_{\lambda\in[a,b]} \sigma(H_0 + \lambda W), \tag{1.1.7}$$

où $W = \sum_{j\in\Gamma} u(x-j)$ est un potentiel périodique.

Étant donnée la famille ergodique de translations $\{\tau_x\}_{x\in T}$ agissant sur Ω, prenons un processus stochastique ergodique $\{Z_x\}_{x\in T}$ sur Ω, i.e., $Z_y(\tau_x(\omega)) = Z_{y-x}(\omega)$, pour tout $y, x \in T$. Le théorème ergodique de Birkhoff assure que si $\mathbb{E}(Z_0) < \infty$, alors

$$\lim_{|\Lambda|\to\infty} \frac{1}{|\Lambda|} \sum_{x\in\Lambda\subset T} Z_x = \mathbb{E}(Z_0), \tag{1.1.8}$$

où $|\Lambda|$ désigne le volume d'un domaine $\Lambda \subset T$ borné. Ce théorème ergodique est utile pour montrer l'existence de la limite de la fonction normalisée de comptage de valeurs propres, inférieures à une certaine énergie $E \in \mathbb{R}$,

$$N(E) := \lim_{|\Lambda|\to\infty} \frac{1}{|\Lambda|} N_\Lambda(E) := \lim_{|\Lambda|\to\infty} \frac{\sharp\{\text{valeurs propres de } H_\omega|_\Lambda \leq E\}}{|\Lambda|}, \tag{1.1.9}$$

où $H_\omega|_\Lambda$ désigne la restriction H_ω au domaine Λ avec des conditions aux bords autoadjointes. Dans le modèle d'Anderson, le terme à volume fini N_Λ dans (1.1.9) est un processus stochastique ergodique. Dans ce cadre, l'existence de la limite dans (1.1.9) peut se voir comme une conséquence du théorème ergodique et de l'*encadrement de Dirichlet–Neumann*,

$$N_\Lambda^D(E) \leq N(E) \leq N_\Lambda^N(E), \tag{1.1.10}$$

où $N_\Lambda^D(E)$ et $N_\Lambda^N(E)$ désignent le terme N_Λ avec des conditions aux bords de Dirichlet et de Neumann, respectivement. On appelle $N(E)$ la densité d'états intégrée et bien que le terme à droite dans (1.1.9) est une quantité aléatoire, la limite N est déterministe (1.1.8). De plus, $N(E)$ est la fonction distribution d'une mesure ν, c'est-à-dire $N(E) = \nu((-\infty, E])$, et le support topologique de la mesure ν est lié au spectre de H_ω par la relation $\sigma(H_\omega) = \operatorname{supp} \nu$ pour presque tout $\omega \in \Omega$.

L'ergodicité, liée elle même à la structure géométrique du milieu considéré, a donc des conséquences importantes pour l'étude spectrale des milieux désordonnés. Cependant, on trouve dans la nature des matériaux ayant des structures géométriques plus exotiques qui ne gardent pas ce trait lorsque l'on les modélise par des OSA.

C'est le cas des *quasi-cristaux*, découverts par Schechtman et al. [Sch84] en 1982 et qui valut à Schechtman le Prix Nobel de chimie en 2011. Ce sont des matériaux qui n'offrent aucune invariance par translations, mais présentent des symétries rotationnelles. Dans ce sens, les quasi-cristaux sont des structures intermédiaires entre des structures cristallines, parfaitement périodiques qui ont du spectre absolument continu, et des structures complètement désordonnées. Un exemple de cette dernière est un ensemble discret de points suivant une distribution de Poisson, qui rentre dans le cadre des OSA et auquel est associé du spectre purement ponctuel.

Une manière mathématique de représenter les quasi-cristaux est d'utiliser des *ensembles de Delone (Delaunay)*. Ce sont des sous-ensembles de l'espace euclidien qui sont *relativement discrets* (il existe une distance minimale r entre les points voisins) et *uniformément denses* (il existe une distance maximale R entre les points voisins). Les ensembles de Delone sont une notion plus générale que les quasi-cristaux, qui exhibent un certain degré d'ordre dans sa structure atomique. Pour mieux modéliser les symétries rotationnelles des quasi-cristaux on attribue aux ensembles de Delone des propriétés géométriques qui décrivent leur degré d'ordre à longue portée. Ces propriétés sont construites autour de la notion de *motif* : un sous-ensemble d'un ensemble de Delone D qui est de la forme $D \cap K$, où $K \subset \mathbb{R}^d$ est compact. On dit que l'ensemble de Delone D possède une *complexité locale finie* si pour tout sous-ensemble compact $A \subset \mathbb{R}^d$, il existe une collection finie $\mathcal{F}_A$ de motifs dans D, telle que tout motif qui est équivalent à $D \cap A$ par translation, i.e., qui est de la forme $D \cap A'$, où A' est un translaté de A, est équivalent par translation à un membre de $\mathcal{F}_A$. En outre, l'ensemble D est *linéairement répétitif* si tout motif dans D est répété dans toute région de taille linéairement proportionnelle à la taille du motif. Un exemple de ce dernier cas sont les sommets d'un pavage de Penrose [Sen] (Fig. 1.1). Plus quantitativement, on peut étudier le taux de répétition des motifs ou même la "fréquence d'apparition des motifs", entre autres. Cependant, ces concepts ne donnent pas une description exhaustive de la nature même de l'ordre apériodique, comme le remarquent Baake, Grimm et Moody dans leur article intitulé "What is aperiodic order ?" [BaGM02]. Cette nature si générale place l'étude de ces ensembles dans un croisement de sujets comme les systèmes dynamiques, la géométrie, la théorie de pavages, la théorie de la diffraction, parmi d'autres (voir [Sen, BaM00] et leurs références). Pour une présentation sur les aspects physiques des quasi-cristaux, en particulier, sur le transport dans des milieux apériodiques, voir la monographie de Bellissard [Bel03].

Nous nous intéressons aux opérateurs définis à partir de telles structures apériodiques et à ses propriétés spectrales. Dans les cas unidimensionnels, notamment dans les travaux de Süto [Su89]

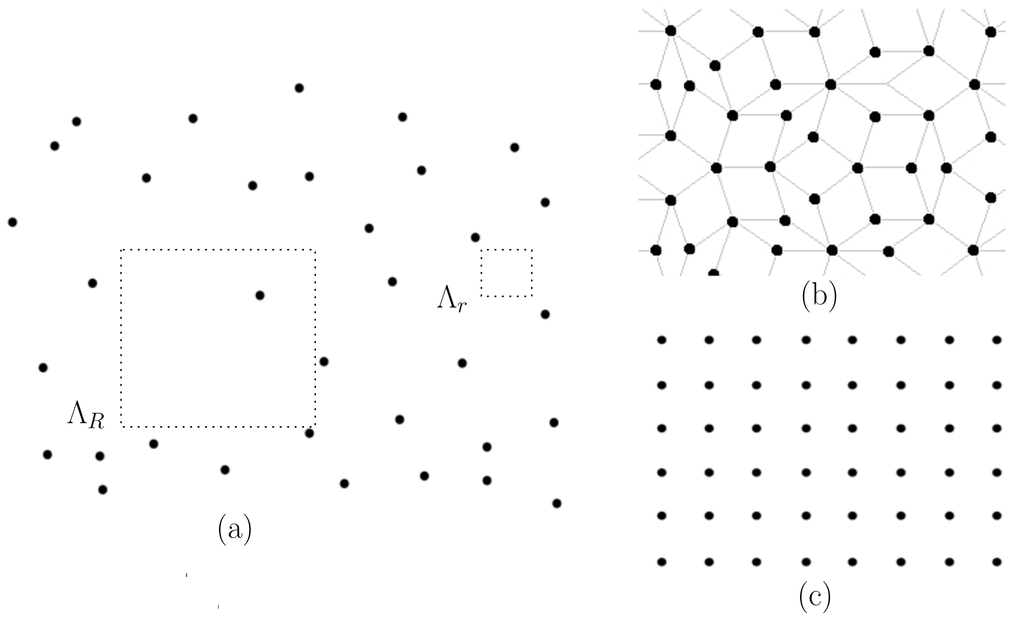

Fig. 1.1 – a) Un ensemble de Delone D est caractérisé par deux paramètres r et R, tels que tout cube de côté r contient *au plus* un élément de D, et tout cube de côté R contient *au moins* un élément de D. (b) Les lignes grises représentent un pavage de Penrose (construit avec des losanges), et l'ensemble de sommets est un exemple d'un ensemble de Delone linéairement répétitif. (c) Un réseau est un exemple d'un ensemble de Delone périodique.

sur le hamiltonien de Fibonacci et ceux de Bellissard et al. [BIST89] sur des quasi-cristaux en général, on trouve un spectre de Cantor purement singulier continu, avec des fonctions propres qui ne sont ni étalées ni localisées. Pour plus de détails, nous conseillons de consulter [D00]. Dans le cas continu en dimension arbitraire, Lenz et Stollmann [LS06] prouvèrent, en utilisant le théorème de "Wonderland" de B. Simon, que, génériquement, les opérateurs de Delone,

$$H_D = -\Delta + \sum_{j\in D} u(x-j), \quad u \in \mathcal{C}_c(\mathbb{R}^d), \tag{1.1.11}$$

ont du spectre purement singulier continu (voir aussi [KLS11]). Pour le Laplacien discret sur un pavage de Penrose la densité d'états intégrée est discontinue [KoSu86, FuKr88] puisqu'il est possible d'avoir des fonctions propres du Laplacien qui sont à support compact. Ensuite Lenz, Stollmann et Klassert [KLS03b] (voir aussi [LV09]) montrèrent que ce comportement, qui peut paraître anormal, est en fait naturel, puisqu'à partir d'un ensemble de Delone quelconque on peut toujours, par une modification locale et sous certaines conditions sur la complexité de l'ensemble, obtenir un ensemble qui permet des fonctions propres à support compact. De plus, celle-ci s'avère être la seule manière de produire une discontinuité dans la densité d'états intégrée.

On peut retrouver l'ergodicité dans ces systèmes apériodiques en considérant la fermeture de l'ensemble de translatés d'un ensemble de Delone D dans $\mathbb{R}^d$,

$$X_D = \overline{\{D + x \,:\, x \in \mathbb{R}^d\}}, \tag{1.1.12}$$

par rapport à une topologie convenable dans l'espace de tous les ensembles de Delone, qui le rend compact. On désigne par α la translation par des éléments de $\mathbb{R}^d$ et on appelle le triplet

$(X_D, \mathbb{R}^d, \alpha)$ un système dynamique de Delone. Ces notions peuvent être formulées dans un cadre plus abstrait mais nous nous restreignons au cas qui nous intéresse. Ces systèmes dynamiques sont étroitement liés aux systèmes dynamiques de pavages. De plus, on peut identifier un ensemble de Delone avec des mesures ponctuelles bornées, qui possèdent un atome sur chaque point de l'ensemble, et étudier de ce point de vue la convergence uniforme des fréquences d'apparition des motifs.

De plus, on peut considérer des systèmes dynamiques de Delone à couleurs aléatoires, ce qui plus tard fera le lien entre les opérateurs de Delone et les OSA. On considère un sous-ensemble borélien $\mathbb{A} \subset \mathbb{R}$, que l'on appelle l'*espace de couleurs*. On associe à chaque point p d'un ensemble $P \in X_D$ une couleur $\omega_p \in \mathbb{A}$. L'espace (produit) des réalisations possibles de la couleur $\omega = (\omega_p)_{p\in P}$ de l'ensemble P est

$$\Omega_P := \bigotimes_{p\in P} \mathbb{A}. \tag{1.1.13}$$

L'ensemble de Delone coloré P^ω est donné par

$$P^\omega := \{(p, \omega_p) : p \in P, \omega \in \Omega_P\}, \tag{1.1.14}$$

et il appartient à la fermeture de l'ensemble de translatés de D^ω [MR12], i.e.,

$$\hat{X}_D := \{P^\omega : P \in X_D, \omega \in \Omega_P\} = \overline{\{x + D^\omega : x \in \mathbb{R}^d\}}. \tag{1.1.15}$$

Ainsi, on peut considérer l'opérateur H_ω comme l'application

$$\hat{X}_D \ni P^\omega \mapsto H_\omega := H_0 + \sum_{p\in P} \omega_p u(x-p). \tag{1.1.16}$$

Dans ce contexte, le fait d'avoir un théorème ergodique pour des systèmes dynamiques de Delone joue un rôle fondamental dans la définition des quantités comme la densité d'états intégrée pour un opérateur de Delone, lié à la fréquence d'apparition des motifs qui composent un ensemble. Hof [Hof98] étudia la percolation sur un pavage de Penrose, en prenant des ensembles de Delone colorés avec la propriété de complexité locale finie, et dans ce cadre il construisit des mesures ergodiques pour le système dynamique associé. La condition sur la complexité de l'ensemble de Delone permet de traiter seulement un nombre fini de couleurs. Par ailleurs, [L09] obtint un théorème ergodique pour des translations de mesures bornées sans supposer que l'ensemble de Delone où elles ont du support ait la propriété de complexité locale finie. Müller et Richard [MR12] s'appuyèrent sur les résultats de Hof et Lenz et obtinrent un théorème ergodique pour des ensembles de Delone colorés permettant un ensemble de couleurs continu. Grâce à cette dernière amélioration, ceci devient l'outil le plus approprié pour notre travail lorsque l'on considérera des variables aléatoires régulières. Le rapport entre les propriétés géométriques d'un ensemble de Delone et les propriétés ergodiques des systèmes dynamiques associés, théorèmes ergodiques pour ce cadre y compris, furent largement analysées dans la littérature, [MR12, LP03, KLS03a, LS03].

Les quasi-cristaux sont à l'origine de notre motivation pour l'étude des systèmes non ergodiques. Lorsque l'on considère un modèle d'Anderson où le milieu est un quasi-cristal, il est représenté par un potentiel externe de la forme

$$V_\omega = \sum_{j\in D} \omega_j u(x-j), \quad u \in \mathcal{C}_c(\mathbb{R}^d), \tag{1.1.17}$$

où D est un ensemble de Delone représentant la structure géométrique des atomes dans le quasi-cristal. Dès que D est apériodique, c'est à dire, qu'il n'est pas invariant par translation, l'opérateur $H_\omega = -\Delta + V_\omega$, dit de Delone–Anderson, ne vérifie plus la condition (1.1.3) et donc le modèle est non ergodique. Ceci dit, au moment de définir la densité d'états intégrée nous utiliserons la construction (1.1.15) pour retrouver l'ergodicité, sous certaines conditions géométriques de l'ensemble de Delone D de base.

Du point de vue mathématique, on distingue trois notions de localisation : spectrale, où le spectre de H_ω est purement ponctuel ; celle dit d'Anderson, où on trouve du spectre purement ponctuel avec des fonctions propres à décroissance exponentielle ; et la localisation dynamique, où les moments des paquets d'ondes, initialement localisés en espace et énergie, restent localisés en temps sous l'action de l'opérateur d'évolution de Schrödinger engendré par H_ω. Cette dernière est la notion la plus forte de localisation électronique.

Les premiers résultats rigoureux furent donnés dans les années 70 par Goldsheid, Molchanov et Pastur [GMP77], sur l'étude de la localisation d'Anderson pour un modèle continu à unidimensionnel. En dimension 1, on peut montrer que tout le spectre d'un OSA est localisé. Pour étudier le problème en dimension arbitraire, Fröhlich et Spencer [FS83] développèrent une analyse multi-échelles (MSA, de ses sigles en anglais) pour montrer l'absence de diffusion dans le modèle d'Anderson (discret), qui est basée sur des estimations de la résolvante d'un opérateur . Cette méthode eut diverses améliorations au cours de ces trente dernières années : Martinelli et Scoppola [MS85] montrèrent l'absence du spectre absolument continu, plus tard [DLS85, FMSS85, SW86] montrèrent que le spectre est purement ponctuel avec des fonctions propres exponentiellement décroissantes. Après, von Dreifus et Klein [vDK89] simplifièrent cette procédure qui a pour principale conséquence la localisation d'Anderson. Les développements postérieurs de De Bièvre-Germinet [GdB98] et Damanik-Stollmann [DS01] montrent que la MSA implique la localisation dynamique forte. Pour des mesures de probabilité régulières, la version plus raffinée de cette méthode est celle du Bootstrap MSA de Germinet–Klein [GK01]. Cette analyse, basée sur quatre autres MSA, permet de montrer en outre la localisation semi-uniforme des fonctions propres et la décroissance sous-exponentielle du noyau (d'opérateur) de l'opérateur dévolution en espérance.

Le cas des mesures singulières comme celles du type Bernoulli fut pendant longtemps un problème ouvert, et ne fut résolu qu'en 2005 par le travail remarquable de Bourgain et Kenig [BoK05] portant sur la localisation d'Anderson pour le modèle de Bernoulli. Ils reformulèrent la MSA pour utiliser une estimée de Wegner échelle par échelle et non pas *a priori* vérifiée pour toutes les échelles comme dans le cas usuel (voir la définition 1.2.2). Germinet et Klein [GK11] adaptèrent les idées de Bourgain–Kenig et développèrent une MSA qui démontre de la localisation dynamique pour un grand nombre de modèle, avec des mesures de probabilité non dégénérées, y compris les mesures de Bernoulli.

Dans ce travail nous nous interrogeons sur le transport des électrons du point de vue dynamique. Nous verrons comment l'absence de l'ergodicité peut être surmontée, et comment les méthodes d'analyse usuelles dans les OSA (analyse multi-échelles, estimées de Wegner, parmi d'autres) doivent être modifiées pour convenir à ce cadre et sous quelles contraintes. Nous développons des outils généraux issus de la situation ergodique qui, adaptés à notre cadre, nous permettent de traiter non seulement les opérateurs dits de Delone–Anderson (1.1.17), mais encore des modèles d'Anderson où les variables aléatoires ne sont pas identiquement distribuées. Des exemples de ce dernier cas sont les modèles lacunaires, les modèles avec des variables aléatoires à densités décroissantes, ou bien les modèles aux potentiels surfaciques (voir [BoeKS05, BdMSS05, BdMS03, KV02b, S, FGKM07]).

Pour le cas particulier des modèles de Delone–Anderson, nous obtenons des dépendances explicites des paramètres de l'ensemble de Delone sous-jacent. Ceci dit, pour les propriétés qui n'ont pas de rapport avec la densité d'états intégrée nous ne faisons aucune supposition sur la géométrie de ces ensembles.

Pour décrire la dynamique d'une particule, on considère le moment aléatoire d'ordre $p \geq 0$ au temps t de l'évolution temporelle dans la norme de Hilbert–Schmidt, initialement localisé en espace sur une boite unité centrée au point $u \in \mathbb{Z}^d$ et localisée en énergie par une fonction $\mathcal{X}$ appartenant à $C^\infty_{c,+}(\mathbb{R})$, l'espace des fonctions non-negatives, à support compact dans $\mathbb{R}$ et infiniment différentiables, i.e.,

$$M_{u,\omega}(p,\mathcal{X},t) = \| \langle X-u \rangle^{p/2} e^{-itH_\omega} \mathcal{X}(H_\omega)\chi_u \|_2^2, \tag{1.1.18}$$

dont la moyenne temporelle est donnée par

$$\mathcal{M}_{u,\omega}(p,\mathcal{X},T) = \frac{2}{T}\int_0^\infty e^{-2t/T} M_{u,\omega}(p,\mathcal{X},t)dt. \tag{1.1.19}$$

Dans le cas ergodique, il suffit de prendre $u = 0$, puisque la probabilité de ces événements est invariante par translation. L'opérateur H_ω présente de la *localisation dynamique* forte dans la norme Hilbert–Schmidt dans un intervalle ouvert I si pour tout $\mathcal{X} \in \mathcal{C}^\infty_{c,+}(I)$ on a

$$\sup_{u\in\mathbb{Z}^2} \mathbb{E}\{\sup_{t\in\mathbb{R}} M_{u,\omega}(p,\mathcal{X},t)\} < \infty \quad \text{pour tout } p \geq 0. \tag{1.1.20}$$

On dit que H_ω présente de la localisation dynamique forte dans la norme Hilbert–Schmidt dans une énergie E s'il existe un intervalle ouvert $I \subset \mathbb{R}$ qui contient E, tel que pour tout $\mathcal{X} \in \mathcal{C}^\infty_{c,+}(I)$ l'expression (1.1.20) est vraie. On désigne par Σ_{SI} l'ensemble

$$\Sigma_{SI} = \{E \in \mathbb{R} : H_\omega \text{ présente de la localisation dynamique forte dans } E\} \tag{1.1.21}$$

Dans le cadre ergodique, en absence d'un champ magnétique, en dimension $d \geq 3$ on s'attend à ce que le spectre d'un opérateur de Schrödinger aléatoire présente une transition telle que l'on peut identifier deux régimes différents : la région isolante, caractérisée par les états localisés et la région métallique, caractérisée par les états étalés. On appelle le passage de l'un à l'autre *transition métal-isolant d'Anderson*. En termes spectraux, ce phénomène se traduit par une transition du spectre purement ponctuel au spectre absolument continu. Si une telle transition a pu être observée sur des arbres [K98, AW11], il en va différemment en dimension finie. Sur $\mathbb{Z}^d$ ou $\mathbb{R}^d$, une telle transition semble toujours hors de portée d'une preuve mathématique.

Cependant, une approche dynamique, plus près de l'intuition physique, a été introduite par Germinet et Klein dans [GK04] pour expliquer cette transition. Ils considèrent un exposant du transport local $\beta(E)$ pour mesurer la propagation temporelle des paquets d'ondes initialement localisés en espace et en énergie qui évoluent sous l'action d'un opérateur aléatoire. Ils obtinrent une caractérisation complète de la transition métal-isolant et de l'énergie de mobilité, où la transition a lieu et qui s'avère être un point de discontinuité de β. Dans cette caractérisation de la transition du transport, l'outil principal est l'*analyse multi-échelles Bootstrap* [GK01], où la localisation dynamique forte qu'elle entraîne est utilisée pour caractériser l'ensemble des énergies dont l'exposant de transport β est nul.

Dans le cas magnétique en dimension 2, le hamiltonien de Landau aléatoire représente une particule en mouvement dans un milieu désordonné bidimensionnel soumis à l'influence d'un champ magnétique constant, fort, transversal au plan. Germinet, Klein et Schenker [GKS07] donnèrent la première preuve mathématique rigoureuse de l'existence d'une transition d'Anderson pour ce modèle en s'appuyant sur l'analyse multi-échelles pour exploiter des propriétés de *l'effet Hall quantique entier*. Pour introduire ce phénomène, considérons un conducteur bidimensionnel dans le plan XY où on applique un champ magnétique transversal d'intensité constante B. En appliquant un champ électrique dans la direction x on crée un courant $\vec{J}$ dans la direction x, tandis que dans la direction y les électrons sont soumis à la force de Lorentz, générant ainsi un champ électrique $\vec{E}$ dans cette direction. En équilibre, la force totale qui agit sur l'électron s'annule, donnant la relation entre le courant et le champ électrique $\vec{J} = \sigma \vec{E}$. La matrice 2×2, σ, appelée tenseur de conductivité, a des termes diagonaux nuls et en dehors de la diagonale, des termes $\pm\sigma_H$, où

$$\sigma_H = \nu \frac{q^2}{h}, \quad \nu = \frac{\rho h}{qB}, \tag{1.1.22}$$

que l'on appelle la *conductivité de Hall*, où q est la charge de l'électron, ρ est la densité électronique du matériel, h la constante de Planck et ν est appelé le facteur de remplissage ("filling factor"). On voit que ν est une fonction de la densité ρ et du champ magnétique B, donc en changeant l'intensité de B on s'attendrait à ce que σ_H change aussi continument. En 1980, Klitzing et al. [Kli80] trouvèrent que la conductivité de Hall σ_H à température très basse (et donc où les effets dissipatifs son négligeables) comme fonction du champ magnétique B, présente des *plateaux* où ses valeurs peuvent être calculées avec une excellente précision :

$$\sigma_H = \frac{q^2}{h}\mathbb{N}. \tag{1.1.23}$$

On appelle le phénomène de quantification de la conductance l'effet de Hall quantique entier, et il valut à Klitzing le Prix Nobel de physique en 1985. Suite au travail de Laughlin [L81], Thouless [Th81] et Halperin [Hal82] conjecturèrent que les plateaux de la conductivité de Hall sont dûs à la localisation, qui est l'effet prépondérant à basse température. Plus tard, Kunz [Ku87] montra que pour un désordre suffisamment petit pour qu'il y existe des lacunes spectrales entre les niveaux de Landau, la quantification de σ_H est invariante par rapport au désordre. En effet, la quantification de la conductivité de Hall n'est pas présente dans le cas d'un électron libre dans le plan sous l'effet d'un champ magnétique fort. Dans ce cas la conductivité de Hall suit un comportement plus près du cas classique, mais le phénomène de sa quantification émerge quand l'électron est dans un milieu avec des impuretés, comme l'expliquèrent Bellissard, Schulz-Baldes et van Elst dans [BSvE94]. Plus précisement, pour avoir l'effet de Hall quantique, il est nécessaire d'avoir l'existence d'états localisés entre les niveaux de Landau. Ce phénomène s'avéra universel dans le sens qu'il est invariant par rapport à la géométrie et la nature du matériel considéré, dès que l'expérience vérifie les conditions nécessaires (basse température, champ magnétique fort, et présence d'impuretés dans le matériel), voir [BSvE94, Bel03] et leurs références. De plus, la précision avec laquelle on peut calculer σ_H expérimentalement l'ont rendu très utile en métrologie, en particulier pour définir l'unité de mesure pour la résistance (ohm) [JJ05].

1.2 Présentation des travaux de thèse

1.2.1 L'analyse multi-échelles dans le cadre non ergodique

On considère un opérateur de Schrödinger aléatoire de la forme

$$H_\omega = H_0 + \lambda V_\omega \quad \text{sur} \;\; \mathrm{L}^2(\mathbb{R}^d), \tag{1.2.1}$$

où H_0 est le hamiltonien libre, et λ est un paramètre de désordre que l'on considérera dorénavant être fixe. Le potentiel aléatoire V_ω est l'opérateur de multiplication par la fonction V_ω, qui est telle que $\{V_\omega(x) : x \in \mathbb{R}^d\}$ est un processus stochastique mesurable à valeurs réelles sur un espace de probabilité complet $(\Omega, \mathcal{F}, \mathbb{P})$. On suppose, d'une part, que V_ω est décomposable en une partie non négative appartenant à $\mathrm{L}^1_{\mathrm{loc}}(\mathbb{R}^d)$ et une partie négative relativement bornée en forme par rapport à H_0, avec borne relative inférieure à 1. Et d'autre part, on suppose qu'il existe une constante $\rho > 0$ telle que pour toute paire de boréliens $B_1, B_2 \subset \mathbb{R}^d$ avec $\mathrm{dist}(B_1, B_2) > \varrho$, les processus $\{V_\omega(x) : x \in B_1\}$ et $\{V_\omega(x) : x \in B_2\}$ sont indépendants. On a donc un opérateur auto-adjoint semi-borné inférieurement pour $\mathbb{P}$-p.t. ω. De plus, l'application $\omega \mapsto H_\omega$ est mesurable presque partout et on désigne le spectre de H_ω par σ_ω.

On considère des versions à volume fini de H_ω, restreint à un cube $\Lambda_L(x)$ de centre x et côté L, avec des conditions au bord autoadjointes, que l'on désigne par $H_{\omega,x,L}$ agissant sur $\mathrm{L}^2(\Lambda_L(x))$. On écrit $R_{\omega,x,L}(z) = (H_{\omega,x,L} - z)^{-1}$ pour l'opérateur résolvante et on définit les projections spectrales comme $P_\omega(J) = \chi_J(H_\omega)$ et $P_{\omega,x,L}(J) = \chi_J(H_{\omega,x,L})$ pour un ensemble de Borel $J \subset \mathbb{R}$.

Definition 1.2.1. On dit que l'opérateur H_ω vérifie une estimée de Wegner uniforme avec un exposant de Hölder s dans un intervalle ouvert $\mathcal{J}$ si pour chaque $E \in \mathcal{J}$ il existe une constante Q_W, localement bornée, et $0 < s \leq 1$ tels que

$$\sup_{x \in \mathbb{R}^d} \mathbb{E}\{\mathrm{tr}(P_{\omega,x,L}(E-\eta, E+\eta))\} \leq Q_W \eta^s L^d, \tag{1.2.2}$$

pour tout $\eta > 0$ et $L \in 2\mathbb{N}$. On dit qu'il vérifie une estimée de Wegner uniforme dans une énergie E s'il vérifie (1.2.2) dans un voisinage (ouvert) de E.

La preuve de localisation dans des systèmes désordonnés se construit autour de la décroissance de la fonction de Green, le noyau intégral de l'opérateur résolvante. Pour rendre le problème plus maniable, on considère des restrictions d'opérateurs à un volume fini, à savoir un cube Λ_L de taille L^d, pour ensuite prendre la limite quand le volume tend ver l'infini. On travaille ainsi avec un spectre discret qui, dans la limite, approche le spectre de l'opérateur original. A l'intérieur de ce domaine Λ_L, et par la nature de l'opérateur aléatoire H_ω, les événements correspondant à des restrictions aux cubes plus petits Λ_l, $l << L$ qui sont disjoints, sont indépendants les uns des autres. Cependant, il peut arriver que les spectres des opérateurs restreints à ces cubes soient très proches du spectre de l'opérateur $H_\omega|_{\Lambda_L}$. Puisque les résolvantes deviennent non bornées, ce phénomène, appelé *résonance*, est analogue au problème des petits dénominateurs qui apparaît dans la théorie de perturbation classique des systèmes dynamiques, entre autres, et qui fût résolu par la théorie de Kolmogorov–Arnold–Moser (KAM).

Pour gérer l'apparition des résonances, Fröhlich et Spencer [FS83] développèrent une analyse multi-échelles, en prenant des idées de la théorie de KAM, et de la renormalisation de groupe.

D'un côté, cette méthode a une composante déterministe, qui utilise à la fois des estimées de Combes–Thomas pour obtenir la décroissance de résolvantes locales, mais aussi l'identité géométrique de la résolvante, pour lier des estimées locales sur des cubes différents. D'autre part, la composante probabiliste de cette méthode se sert de l'estimée de Wegner pour contrôler la contribution des régions résonantes, en lui attribuant de faibles probabilités qui n'affectent pas le résultat final. Le but est d'assurer la décroissance des résolvantes locales avec une bonne probabilité. De cette manière, on peut faire une itération par échelles à partir d'une estimée d'échelle initiale L_0, et propager la décroissance de la résolvante locale sur toute une suite d'échelles $L_{k+1} = (L_k)^\alpha$, $0 < \alpha < 1$, avec une bonne probabilité.

L'analyse multi-échelles est peu sensible à la géométrie des configurations d'impuretés sous-jacentes, étant donné une certaine homogénéité dans la distribution des impuretés dans l'espace. Dans les modèles de Delone–Anderson (1.1.17), définis sur un ensemble de Delone de caractéristiques (r, R), malgré le manque d'ergodicité (dans le cas apériodique), cette homogénéité est assurée par le paramètre R qui contrôle la taille maximale des régions sans obstacles dans l'espace. Dans la suite, nous formulerons plus précisement les critères d'uniformité que les modèles doivent vérifier pour permettre d'appliquer l'analyse multi-échelles pour des structures non ergodiques.

Étant donnés $\theta > 0$, $E \in \mathbb{R}$, $x \in \mathbb{Z}^d$ et $L \in 6\mathbb{N}$, on dit que la boite $\Lambda_L(x)$ est (θ, E)-*convenable* pour H_ω si $E \notin \sigma_{\omega,x,L}$ et

$$\|\Gamma_{x,L} R_{\omega,x,L}(E) \chi_{x,L/3}\|_{x,L} \leq \frac{1}{L^\theta},$$

où $\Gamma_{x,L} = \chi_{\bar{\Lambda}_{L-1}(x) \setminus \Lambda_{L-3}(x)}$.

Nous montrons que la Bootstrap MSA admet des modifications permettant l'étude de modèles non ergodiques. Le théorème suivant est une reformulation du Théorème 3.4 et du Corollaire 3.10 [GK01] dans un cadre non ergodique,

Théorème 1.2.1. *Soit H_ω un opérateur de Schrödinger dans le sens défini auparavant vérifiant une estimée de Wegner uniforme dans un intervalle ouvert $\mathcal{J}$ avec un exposant de Hölder s. Étant donné $\theta > d$, pour chaque $E \in \mathcal{J}$ il existe une échelle finie $\mathcal{L}_\theta(E) = \mathcal{L}(\theta, E, Q_W, d, s)$, bornée sur des sous-intervalles compactes de $\mathcal{J}$, telle que si pour $\mathcal{L} > \mathcal{L}_\theta(E)$ on a*

$$\inf_{x \in \mathbb{Z}^d} \mathbb{P}\{\Lambda_{\mathcal{L}}(x) \text{ est } (\theta, E)\text{-convenable}\} > 1 - \frac{1}{841^d}, \tag{1.2.3}$$

alors il existe $\delta_0 > 0$ et $C_\zeta > 0$ tels que

$$\sup_{u \in \mathbb{Z}^d} \mathbb{E}\left(\sup_{\|f\| \leq 1} \|\chi_{x+u} f(H_\omega) P_\omega(I(\delta_0)) \chi_u\|_2^2 \right) \leq C_\zeta e^{-|x|^\zeta}, \tag{1.2.4}$$

pour $0 < \zeta < 1$, où $I(\delta_0) = [E - \delta_0, E + \delta_0]$. Comme conséquence, on a $E \in \Sigma_{SI}$ (voir (1.1.21)*) et on obtient les propriétés suivantes,*

(SUDEC) Décroissance uniforme et sommable des correlations des fonctions propres *(Summable uniform decay of eigenfunction correlations) : pour presque tout $\omega \in \Omega$, le spectre du hamiltonien H_ω dans $I \subset \Sigma_{SI}$ est purement ponctuel de multiplicité finie. Soit $\{\epsilon_{n,\omega}\}_{n \in \mathbb{N}}$ une numération des valeurs propres différentes de H_ω dans I. Alors pour chaque $\zeta \in]0, 1[$*

et $\epsilon > 0$ on a, pour $x, u \in \mathbb{Z}^d$,

$$\|\chi_{x+u}\phi\|\|\chi_u\varphi\| \leq C_{I,\zeta,\epsilon,\omega}\|T_u^{-1}\phi\|\|T_u^{-1}\varphi\|\langle x+u\rangle^{\frac{d+\epsilon}{2}}\langle u\rangle^{\frac{d+\epsilon}{2}}e^{-|x|^\zeta}, \tag{1.2.5}$$

pour tout $\phi, \varphi \in$ Ran $P_\omega(\{\epsilon_{n,\omega}\})$ (voir Section 2.2).

(SULE) Fonctions propres semi-uniformément localisées *(semi-uniformly localized eigenfunctions) : Soit $I \subset \Sigma_{SI}$ et soit $\{\epsilon_{n,\omega}\}_{n\in\mathbb{N}}$ la numération des valeurs propres différentes de H_ω dans I. Pour tout $\epsilon > 0$, il existe une constante $m_\epsilon > 0$ et, pour presque tout $\omega \in \Omega$, il existe une constante $C_{\epsilon,\omega} < \infty$, telles que si $(\varphi_{n,\omega})_{n\in\mathbb{N}}$ sont les fonctions propres normalisées associées aux valeurs propres $\{\epsilon_{n,\omega}\}_{n\in\mathbb{N}}$ dans I, alors il existe des* centres de localisation *$\{x_{n,\omega}\}_{n\in\mathbb{N}}$, tels que pour tout $n \in \mathbb{N}$ et $x \in \mathbb{Z}^d$ on a*

$$\|\chi_x\varphi_{n,\omega}\| \leq C_{\epsilon,\omega}e^{m_\epsilon(\log|x_{n,\omega}|)^{1+\epsilon}}e^{-m_\epsilon|x-x_{n,\omega}|}. \tag{1.2.6}$$

De plus, les centres de localisation $x_{n,\omega}$ peuvent s'ordonner de telle manière que

$$|x_{n,\omega}| \geq \tilde{C}_\omega n^{1/(4\nu)}, \tag{1.2.7}$$

pour une constante finie $\tilde{C}_\omega > 0$, et la constante $\nu > d/4$ comme dans la propriété UGEE, voir (2.2.8).

(DFP) Décroissance des projections de Fermi *(Decay of the Fermi projections) : pour $E \in \Sigma_{SI}$ et pour $\zeta \in]0,1[$ quelconque on a*

$$\sup_{u\in\mathbb{Z}^d}\mathbb{E}\{\|\chi_{x+u}P_\omega((-\infty,E])\chi_u\|_2^2\} \leq C_{\zeta,\lambda,E}e^{-|x|^\zeta}, \tag{1.2.8}$$

où la constante $C_{\zeta,E}$ est localement bornée en E.

L'ensemble d'énergies où on peut démarrer le Théorème 2.1.3, c'est-à-dire, là où H_ω vérifie une estimée de Wegner uniforme (1.2.2) ainsi comme l'estimé de pas initial (1.2.3) est désigné par Σ_{MSA}.

Dans notre démonstration, nous formulons la procédure de récurrence de l'analyse multi-échelles de manière locale, avec des boîtes centrées sur des points arbitraire de l'espace. De même, nous construisons localement une expansion en fonctions propres généralisées, qui fera le passage entre l'analyse multi-èchelles et la localisation dynamique. Nous montrons ensuite que, pour des modèles assez généraux, sous la contrainte d'uniformité des estimées de Wegner et l'estimée d'échelle initiale, la MSA adaptée localement donne les mêmes résultats que dans le cas ergodique.

1.2.2 Transition métal-isolant d'Anderson

Nous étudions le régime de localisation dynamique en utilisant l'exposant de transport $\beta(E)$ défini comme le taux de croissance de la fonction $\mathcal{M}_{u,\omega}(p,\mathcal{X},T)$, définie par (1.1.19), dans l'expression

$$\sup_u \mathbb{E}\left(\mathcal{M}_{u,\omega}(p,\mathcal{X},T)\right) \sim T^{p\beta(E)}. \tag{1.2.9}$$

On définit deux ensembles complémentaires : la région de localisation dynamique $\Xi^{DL} = \{E \in \mathbb{R} : \beta(E) = 0\}$ et celle de délocalisation dynamique $\Xi^{DD} = \{E \in \mathbb{R} : \beta(E) > 0\}$. Ces régions sont respectivement appelées dans la littérature “région de transport métallique faible” et “région de délocalisation dynamique ou de transport trivial”. Par définition, la région de localisaton dynamique Σ_{SI}, définie par (1.1.20), est contenue dans Ξ_{DL} qui, donc par le Théorème 1.2.1, contient la région Σ_{MSA}, d'applicabilité de la MSA. Ce qui complète la caractérisation de la région de localisation dynamique, est le fait que si β est au dessous de la valeur critique $s/2d$, alors on peut obtenir le pas initial (1.2.3) pour démarrer la MSA. Pour cela, nous obtenons [GK04, Theorem 2.11] dans le cas non ergodique :

Théorème 1.2.2. *Soit H_ω un opérateur de Schrödinger aléatoire comme il est défini auparavant, qui vérifie une estimée de Wegner avec un exposant de Hölder s dans un intervalle ouvert $\mathcal{J}$. Soit $\mathcal{X} \in C^\infty_{c,+}(\mathbb{R})$ avec $\mathcal{X} \equiv 1$ sur un intervalle $J \subset \mathcal{J}$, $\alpha \geq 0$ et $p > p(\alpha, s) := 12\frac{d}{s} + 2\alpha\frac{d}{s}$. Si*

$$\liminf_{T\to\infty} \sup_{u\in\mathbb{Z}^d} \frac{1}{T^\alpha}\mathbb{E}\left(\mathcal{M}_{u,\omega}(p, \mathcal{X}, T)\right) < \infty, \tag{1.2.10}$$

alors $J \subset \Sigma_{MSA}$. En particulier, (1.2.10) *est vérifié pour tout $p \geq 0$.*

Le Théorème 1.2.2 implique la relation $\Xi^{DL} \subset \Sigma_{MSA}$, et comme conséquence, que l'ensemble d'énergies dont l'exposant β est nul est équivalent à l'ensemble où on a de la localisation dynamique, i.e., $\Xi^{DL} = \Sigma_{SI}$, et plus, $\Xi^{DD} = \{E \in \mathbb{R} : \beta(E) > s/2d\}$. L'énergie de mobilité qui sépare ces deux régions de localisation et délocalisation dynamiques est un point de discontinuité de l'exposant β. Ainsi, l'exposant de transport $\beta(E)$ donne une caractérisation de la transition de transport métal-isolant pour des modèles non-ergodiques, ce qui est connu dans le cadre ergodique. Ensuite, nous nous intéressons à la validité du résultat du théorème précédent si l'information sur la croissance du moment aléatoire $\mathcal{M}_{u,\omega}(p, \mathcal{X}, T)$ est donnée juste en probabilité. Ceci donne le résultat dans un version presque sûre (régime *quenched*) puisque l'on connaît son comportement pour un ensemble de réalisations du potentiel aléatoire, et non plus en moyenne sur l'aléa qui est le régime habituel (*annealed*).

Théorème 1.2.3. *Soit H_ω l'opérateur de Schrödinger aléatoire défini auparavant, qui vérifie une estimée de Wegner avec exposant de Hölder s dans un intervalle ouvert $\mathcal{J}$. Soit $\mathcal{X} \in C^\infty_{c,+}(\mathbb{R})$ avec $\mathcal{X} \equiv 1$ sur un intervalle $J \subset \mathcal{J}$, $\alpha \geq 0$ et $p > p(\alpha, s) := 15\frac{d}{s} + 2\alpha\frac{d}{s}$. Si*

$$\liminf_{T\to\infty} \sup_{u\in\mathbb{Z}^d} T^{\frac{s}{d}}\mathbb{P}(\mathcal{M}_{u,\omega}(p, \mathcal{X}, T) > T^\alpha) = 0, \tag{1.2.11}$$

alors $J \subset \Sigma_{MSA}$. En particulier, (1.2.11) *est vérifié pour tout $p \geq 0$.*

Remarque 1.2.2. *Si le moment aléatoire croît presque sûrement moins vite qu'un polynôme, ceci implique en particulier la condition* (1.2.11) *pour un certain $\alpha > 0$ et on a les conséquences du théorème.*

De plus, si la condition (2.28) dans [GK04, Theorem 2.11] est vraie pour $\alpha > 0$ et $p > p(\alpha, s) + d$, alors la condition (1.2.11) *est vérifiée pour $\alpha' = \alpha + \delta$ et le même p, où $0 < s/2 < \delta < \frac{s(p-p(\alpha,s))}{2d}$ et $p > p(\alpha', s)$, puisque par l'inégalité de Chebyshev on a, pour tout $T > 0$*

$$T^{\frac{s}{d}} \sup_u \mathbb{P}(\mathcal{M}_{u,\omega}(p, \mathcal{X}, T) > T^{\alpha'}) \leq \frac{1}{T^{\alpha+\delta-s/2}} \sup_u \mathbb{E}(\mathcal{M}_{u,\omega}(p, \mathcal{X}, T)). \tag{1.2.12}$$

Donc (1.2.11) *est certainement une condition plus faible que* (1.2.10).

Nous voyons ainsi que la dynamique de la solution de l'équation de Schrödinger $\psi(t,x) = e^{-itH_\omega}\psi(0,x)$, dans (1.1.1), est semblable dans sa version moyennée (annealed) et presque sûre (quenched).

Pour illustrer que ceci n'est pas toujours le cas, considérons le modèle d'Anderson parabolique (MAP) [GaMo90, GaKo05, GaKoMo07]. Ceci est le problème de Cauchy pour l'équation de la chaleur dans $\mathbb{Z}^d$ avec un potentiel aléatoire, donné par

$$\begin{aligned} \partial_t u(t,x) &= \Delta u(t,x) + \xi(x)u(t,x) \quad \text{on } \mathbb{R}_+ \times \mathbb{Z}^d \\ u(0,x) &= \delta_0(x) \end{aligned} \tag{1.2.13}$$

où Δ est le Laplacien discret et ξ est le potentiel ergodique à valeurs réelles avec des variables aléatoires identiquement distribuées. Le comportement asymptotique en temps de la solution u de (1.2.13) est déterminé par les propriétés spectrales de l'opérateur $H = \Delta + \xi$. Ce modèle est important en physique mathématique puisqu'on observe de *l'intermittence*. Cela signifie qu'il existe un nombre petit de régions dans l'espace, appelées des îles d'intermittence, qui sont loin les unes des autres et qui portent asymptotiquement toute la *masse* de la solution u, définie par la quantité aléatoire $U(t) = \sum_{x\in\mathbb{Z}^d} u(t,x)$. Pour obtenir une caractérisation géométrique du phénomène d'intermittence, on cherche à comprendre le comportement asymptotique de $U(t)$ lorsque $t \to \infty$, la géométrie des îles d'intermittence qui portent la plus grande partie de la masse $U(t)$, ainsi que la forme typique du potentiel ξ et la forme de la solution u sur ses îles. Le régime moyenné correspond à l'étude de l'espérance de $U(t)$ dans l'espace de probabilité, c'est à dire, sur l'espace de toutes les configurations possibles du potentiel ξ, comme dans les OSA, alors que dans le régime presque sûr on regarde les asymptotiques pour un ensemble de réalisations de mesure totale dans l'espace de probabilités. On observe des comportements différents dans ces deux régimes, par rapport à la forme, la taille et la quantité des îles d'intermittence comme à la forme du potentiel ξ qui gouverne ces traits [GaMo90]. La transition du régime presque sûr au régime moyenné fut analysée dans [BAMR05], en considérant les asymptotiques d'une version de la masse $U(t)$ à volume fini, normalisée par le volume.

1.2.3 Opérateurs de Delone–Anderson

Comme application des outils développés précédemment, nous poursuivons avec l'étude des opérateurs de la forme $H_\omega = H_0 + \lambda V_\omega$. La partie libre H_0 est un opérateur auto-adjoint semi-borné inférieurement et V_ω est l'opérateur de Delone–Anderson définit par

$$V_\omega = \sum_{j\in D} \omega_j u(x-j), \quad u \in \mathcal{C}_c(\Lambda_r), \tag{1.2.14}$$

où D est un ensemble de Delone de paramètres (r,R) pas forcément périodique, et où les variables aléatoires ω_j sont indépendantes et identiquement distribuées. On suppose que le potentiel de simple site $u \in \mathcal{C}_c(\Lambda_r)$, l'ensemble de fonctions continues à support compact contenu dans le cube de côté R. Notons que, comme r est la distance minimale entre les points de D, u ne vérifie pas la *condition de recouvrement*

$$\sum_{j\in D} u(x-j) \geq C\chi_{\mathbb{R}^d}, \quad \text{avec } 0 < C < \infty, \tag{1.2.15}$$

puisque les supports de $u(x-j)$ sont disjoints.

Nous cherchons à montrer la localisation dynamique et à obtenir une borne quantitative sur l'intervalle de localisation ainsi qu'à obtenir la transition métal-isolant d'Anderson dans le cas magnétique en dimension $d = 2$. Notre problème consiste à démontrer les deux hypothèses nécessaires pour démarrer l'analyse multi-échelles pour le cadre non ergodique : l'estimée de Wegner (1.2.2) et l'estimée d'échelle initiale (1.2.3).

Obtenir l'estimée de Wegner s'avèra une question difficile en soi, puisque le problème du manque d'invariance par translation dans (1.2.14) se voit empirer par le manque de condition de recouvrement (1.2.15) pour le potentiel de simple site u. Une manière d'obtenir des estimées de Wegner, dans le cadre ergodique, est de "lever" le spectre en exprimant l'opérateur aléatoire comme une perturbation négative d'un opérateur maximal (périodique, dans le cas où D est un réseau) dont l'infimum spectral se situe strictement au-dessus de l'infimum spectral de l'opérateur original. De cette manière l'estimée de Wegner est obtenu en dehors du spectre d'un opérateur périodique [BdMLS11, BdMNSS06].

Dans [BdMNSS06], pour une configuration d'impuretés du type Delone et $H_0 = -\Delta$, on montre que l'infimum spectral de l'opérateur maximal est strictement positif en utilisant des conditions de bord de Neumann et un argument de compacité. Ensuite on utilise la méthode de moments fractionnaires [AM93, A94] pour montrer de la localisation en bas du spectre. Cependant, cette analyse ne donne pas d'information quantitative sur le placement de cet infimum spectral de l'opérateur maximal. On a comme but dans ce travail, entre autres, d'obtenir de l'information sur la taille de la région de localisation, notamment, une borne sur sa dépendance aux paramètres de l'ensemble de Delone.

Dans ce contexte, il est fondamental que l'opérateur libre H_0 vérifie le *principe de continuation unique* (UCP, de ses sigles en anglais). Cela signifie que pour tout $E \in \mathbb{R}$ et pour toute $\varphi \in \mathrm{H}^2_{\mathrm{loc}}(\mathbb{R}^d)$ qui est solution de l'équation $(H_0 - E) = 0$, si φ s'annule sur un ensemble ouvert de $\mathbb{R}^d$, alors φ est identiquement nulle. Pour des potentiels $\mathbf{A}$ et V_ω assez réguliers, l'opérateur vérifie une telle propriété [W95]. Pour illustrer de manière simple la version quantitative de ce principe, regardons une fonction propre φ de l'opérateur H_0 restreint à un volume $G \subset \mathbb{R}^d$. Si $\varphi \neq 0$ sur G, alors pour $x \in \mathbb{R}^d$, $\delta > 0$ et $\Theta \subset \mathbb{R}^d$, tels que $B(x, \delta)$ et Θ sont contenus dans G et loin de ∂G, il existe une constante $C > 0$, dépendante de d, G, Θ et de $\mathrm{dist}(x, \Theta)$, telle que,

$$\|\varphi\|^2_{B(x,\delta)} \geq C \, \|\varphi\|^2_{\Theta} \quad \text{sur } G. \tag{1.2.16}$$

La version quantitative de l'UCP est conséquence des estimées de type Carleman [BoK05, GK11, EV03]. L'UCP fut exploité par Combes, Hislop et Klopp dans [CHK03, CHK07] pour obtenir des estimées de Wegner pour le modèle d'Anderson sans avoir une condition de recouvrement (1.2.15) pour le potentiel de simple site u. L'ingrédient principal de leur preuve est une borne inférieure pour les projecteurs spectraux $P_0(I)$ de l'opérateur H_0 associés à un intervalle $I \subset \mathbb{R}$, de la forme

$$P_0(I) \, V \, P_0(I) \geq C' P(I), \tag{1.2.17}$$

pour une constante $C' > 0$, où V est le potentiel (déterministe) périodique obtenu en faisant toutes les variable aléatoires égales à 1 dans V_ω, (1.1.6), i.e.,

$$V(x) = \sum_{j \in \Gamma} u(x - j). \tag{1.2.18}$$

Comme le potentiel de simple site u vérifie $u \geq u_- \chi_{B(0,\delta)}$ pour des constantes $u_-, \delta > 0$, et δ

petit, (1.2.17) revient à dire que si $\varphi \in \text{Ran } P_0(I)$, alors

$$\sum_{j\in\Gamma} \|\varphi\|^2_{B(j,\delta)} \geq C''\|\varphi\|^2, \tag{1.2.19}$$

pour une constante $C'' > 0$. Le résultat dans [CHK03] est basé sur la périodicité du réseau et l'utilisation de la théorie de Floquet. Connaître la constante C' dans (1.2.17) est importante puisqu'elle rentre dans la constante Q_W de l'estimée de Wegner (1.2.2) comme C'^{-2} [CHK03, CHK07], cependant, sa forme explicite ne fut pas nécessaire dans ces résultats. Par ailleurs, cette information est fondamentale pour Bourgain et Kenig [BoK05] dans leur preuve de localisation d'Anderson pour le modèle de Bernoulli dans $\mathrm{L}^2(\mathbb{R}^d)$. Ils montrèrent que si l'on écrit $R := \text{dist}(x, \Theta)$ et $\tau := \|\varphi\|_G\|\varphi\|_\Theta^{-1}$ dans (1.2.16), la constante C dépend de ces paramètres comme

$$C \approx R^{-R^{4/3}\log\tau}. \tag{1.2.20}$$

Plus tard, Germinet et Klein [GK11] se basèrent sur ces résultats pour donner une formule explicite pour C' dans (1.2.17), en utilisant la théorie de Bloch-Floquet. Une décomposition de Floquet du potentiel (1.2.18) permet de réduire l'étude de φ sur G à l'étude de son comportement sur Λ_q, la cellule élémentaire du réseau Γ, de période q. Si $q \geq 2$, ils obtinrent l'estimée (1.2.16) avec $G = \Lambda_q$, $B(x,\delta) \subset \text{supp}\, u$ et une constante

$$C \approx q^{-q^{4/3}}, \tag{1.2.21}$$

qui ne dépend pas du quotient τ. Comme l'estimée est uniforme par rapport au placement de la cellule dans G, au moment de réunir les termes pour réconstruire Γ, on obtient (1.2.23) avec la même constante C. Ainsi, dans (1.2.23), la constante dépend seulement de la période du réseau et non pas de la taille de φ sur G ni du quotient τ.

Dans le cas où la distribution d'impuretés dans le milieu n'est pas périodique, par exemple, le cas des modèles de Delone–Anderson, la manque d'une théorie de Bloch-Floquet s'avéra un obstacle important. Prenons un opérateur de Delone–Anderson avec un ensemble de Delone sous-jacent apériodique D, tel que l'on peut décomposer G en cellules Λ_q où chaque point j de D est à l'intérieure d'une cellule, que l'on désigne par $\Lambda_q(j)$. Si on essaye de procéder comme dans [GK11], on obtient l'estimée (1.2.16) avec une constante C qui peut être différente pour chaque cellule élémentaire $\Lambda_q(j)$, disons, C_j, et qui peut même dépendre d'un quotient $\tau_j := \|\varphi\|_G\|\varphi\|^{-1}_{\Lambda_q(j)}$. Ensuite, on obtient une estimée équivalente à (1.2.23) sur G, de la forme

$$\sum_{j\in D\cap G} \|\varphi\|^2_{B(j,\delta)} \geq \sum_{j\in D\cap G} C_j\|\varphi\|^2_{\Lambda_q(j)}, \tag{1.2.22}$$

tandis que dans le cas périodique, on a $C_j = C_0$ et $\tau_j = \tau_0\ \forall j \in D$. Or, rien n'assure que $\inf\{C_j,\, j \in D \cap G\}$ soit indépendant de la taille de G et que, donc, le terme à gauche dans (1.2.23) reste positif quand G tend vers tout l'espace $\mathbb{R}^d$. Dans ce cadre, des résultats analogues à ceux de [GK11] furent obtenus dans [RMV12] pour des fonctions propres de l'opérateur H_0, pour $H_\omega = -\Delta + V_0 + V_\omega$, où V_0 est un potentiel borné. La principale difficulté dans le cas apériodique est de contrôler le terme $\|\varphi\|_G\|\varphi\|_\Theta^{-1}$ qui apparaît dans la constante C et ainsi éliminer toute dépendance de la taille de G dans (1.2.23) qui donnerait que $\inf\{C_j,\, j \in D\} > 0$. C'est ce que nous obtenons dans [RMV12], où, par un argument de réduction géométrique, pour $\varphi \in \text{Ran } P_0(I)$ une fonction propre, on a (1.2.23) à volume fini, soit :

$$\sum_{j\in D\cap G} \|\varphi\|^2_{B(j,\delta)} \geq C'' \sum_{j\in D\cap G} \|\varphi\|^2_{\Lambda_q(j)} = C''\|\varphi\|^2_G. \tag{1.2.23}$$

Dans l'Appendice A.1, on présente une preuve simplifié du résultat dans [RMV12]. En bas du spectre, cela entraîne (1.2.17) par un argument de [BdMLS11] et sans avoir recours à une décomposition de Floquet. Pour un intervalle $I \subset \mathbb{R}$ arbitraire, (1.2.17) reste un problème ouvert dans le modèle de Delone–Anderson. En dimension 1, il y a des résultats du type (1.2.23) pour des fonctions propres [K96, V96, KV02a], que l'on considérera plus tard avec le modèle de Delone–Bernoulli.

En ce qui concerne l'estimée d'échelle initiale, cela consiste à établir la décroissance de la résolvante à volume fini à partir d'une certaine échelle. En bas du spectre, ceci est une conséquence de l'existence d'une lacune spectrale pour les versions à volume fini de H_ω, où on peut utiliser l'estimée de Combes–Thomas. Aux bords du spectre, on peut aussi utiliser des asymptotiques de Lifshitz, qui est un comportement caractéristique de la densité d'états intégrée (IDS) sur ces régions spectrales. Cet argument ne fonctionne pas en toute généralité dans le cas d'un ensemble de Delone apériodique, où, faute d'un théorème ergodique convenable, même l'existence de l'IDS s'avère un problème plus délicat que l'on traitera dans le dernier chapitre de cette thèse, et où on trouvera des asymptotiques de Lifshitz seulement en bas du spectre. Les lacunes spectrales internes restent encore hors de portée.

La version quantitative de l'UCP est aussi importante pour avoir de l'information sur l'infimum spectral. Si on connaît la constante C' dans (1.2.17) pour l'état fondamental φ, normalisé, d'un opérateur H_0 et un potentiel V, alors (1.2.17) implique

$$\langle \varphi, (H_0 + V)\varphi \rangle = \langle \varphi, H_0\varphi \rangle + \langle \varphi, V\varphi \rangle \quad (1.2.24)$$
$$\geq E_0 + C', \quad (1.2.25)$$

où $E_0 = \inf \sigma(H_0)$. Le lien entre la perturbation de l'infimum spectral et (1.2.17) fut étudié, notamment, dans [BdMLS11]. Dans ce cadre, ces inégalités deviennent utiles pour estimer la taille des lacunes spectrales en bas du spectre et donc, pour l'obtention de l'estimée d'échelle initiale.

Nous regroupons nos résultats par type d'opérateur non perturbé H_0 : avec puis sans champ magnétique.

1.2.4 Opérateurs de Landau avec perturbation du type Delone–Anderson

Considèrons le hamiltonien de Landau avec champ magnétique constant B et une perturbation du type Delone–Anderson, que l'on dénote par $H_{B,\lambda,\omega}$. Nous appliquons les résultats précédents et on montre l'existence d'une transition métal-isolant, comme dans le cas ergodique [GKS07]. Plus précisement, on montre l'existence des régions spectrales complémentaires de localisation et délocalisation dynamiques. En faisant cela, on généralise des résultats connus pour les hamiltoniens de Landau aléatoires ergodiques [CH96, GK03, GKS07, GKS09] au cadre non ergodique.

Le spectre de $H_{B,\lambda,\omega}$ est contenu dans des bandes, dites de Landau. Si le paramètre de désordre λ et le support des variables aléatoires sont tels que les bandes de Landau sont disjointes, nous montrons

Théorème 1.2.4. *Soit $H_{B,\lambda,\omega}$ un opérateur de Delone-Landau, avec potentiel V_ω donné par* (1.2.14)*, pour lequel les bandes de Landau sont disjointes. Alors pour chaque $n = 0, 1, 2, \ldots$ il existe une constante positive $B(n)$, dépendante des paramètres du modèle, telle que pour tout $B > B(n)$,*

$H_{B,\lambda,\omega}$ présente une transition métal-isolant d'Anderson presque surement dans l'enième bande de Landau.

Par rapport à l'estimée de Wegner, on se base sur [CHKR04], où l'utilisation d'une décomposition de Floquet est remplacée par des estimées sur des projecteurs spectraux. Ces arguments sont convenables dans le cadre apériodique, puisque ces propriétés, inhérentes à la situation magnétique, ne dépendent pas de la périodicité du réseau. L'estimée d'échelle initiale est abordée dans le Théorème 3.3.1, où nous prouvons la localisation dynamique aux bords des bandes de Landau.

La délocalisation dynamique est, par ailleurs, établie à l'intérieur de chaque bande de Landau dans le Théorème 3.3.4. Ceci est une conséquence, d'une part, de la quantification de la conductivité de Hall σ_H (1.1.23) et d'autre part, de la décroissance des projecteurs de Fermi donnée par le Théorème 1.2.1. Les arguments de [GKS07] ne sont pas sensibles à la périodicité du réseau et donc on peut suivre le même raisonnement. Rappelons que la conductivité de Hall σ_H est quantifiée sur des intervalles de localisation dynamique. De plus, sous la condition d'avoir des bandes de Landau disjointes, dans un certain régime du paramètre de désordre λ (petit), les lacunes spectrales restent ouvertes et donc, la valeur de σ_H sur ces régions reste constante. Puisque σ_H a des valeurs dans $\mathbb{N}$ pour le hamiltonien de Landau libre [BSvE94] sur les lacunes spectrales, on peut en déduire que la conductance de Hall ne peut pas rester constante dans tout le spectre et donc il existe au moins une énergie délocalisée. De plus, comme nous avons montré de la localisation dynamique aus bords des bandes spectrales, ceci montre l'existence d'une énergie de mobilité entre les régions métallique et isolante.

De plus, le Théorème 3.3.5 assure que ces résultats ne sont pas triviaux en démontrant qu'il existe presque sûrement du spectre de $H_{B,\lambda,\omega}$ dans les régions où nous démontrons qu'il y a de la localisation dynamique. Plus précisement, nous montrons que s'il y a des lacunes spectrales dans des bandes de Landau, leur taille doit être plus petite que $B^{-1/2}$. Ce résultat fut démontré dans le cadre ergodique dans [CH96, Appendix B]. Pour l'adapter aux opérateurs de Delone–Anderson, nous considérons le système dynamique de Delone colorié $\hat{X}_D$, défini dans (1.1.15). Dans cette construction, il est fondamental d'avoir la bonne définition de translation, à savoir, quand on translate un ensemble $D^\omega \in \hat{X}_D$ dans $\mathbb{R}^d$, on le fait de manière que la variable aléatoire ω_j, $j \in D$ est translatée avec le point $j \in D$. Autrement dit, le translaté du point $(j, \omega_j) \in D^\omega$ est $(j + x, \omega_j) \in (D + x)^\omega$, dans un sens que l'on décrira rigoureusement dans le Chapitre 5. On généralise [CH96, Appendix B] en utilisant une translation particulière de D^ω, et les résultats obtenus sont uniformes par rapport à la translation choisie.

1.2.5 Perturbations du type Delone–Anderson de $H_0 = -\Delta$ et $H_0 = -\Delta + V_0$

On considère des perturbations de type Delone–Anderson d'un opérateur H_0 où $H_0 = -\Delta$ ou bien $H_0 = -\Delta + V_0$, et V_0 est un potentiel déterministe borné. On vérifie les hypothèses du Théorème 1.2.1 pour montrer de la localisation dynamique en bas du spectre.

Dans le cas $H_0 = -\Delta$, en ce qui concerne l'estimée de Wegner pour le Laplacien libre en bas du spectre, nous nous servons d'une méthode de moyennage spatiale comme dans [GHK07, BoK05, G08]. Cela consiste à utiliser un potentiel auxiliaire qui reproduit une condition du recouvrement (1.2.15) et qui, donc, "lève" le spectre et donne le principe de continuation unique quantitative dont on a besoin pour obtenir un estimée de Wegner optimale en suivant les arguments de

[CHK07, CHK03], sans utiliser une décomposition de Floquet. Plus précisement, pour le potentiel V_Λ défini comme la restriction de (1.2.18) sur un cube Λ, nous prenons le potentiel auxiliaire $\overline{V}_\Lambda$,

$$\overline{V}_\Lambda(\cdot) = \frac{1}{|\Lambda|}\int_\Lambda V_\Lambda(\cdot - a)\mathrm{d}a \geq C\chi_\Lambda(\cdot), \tag{1.2.26}$$

où la constante C dépend de u, R, d. On peut montrer que pour $\varphi \in \operatorname{Ran} P_{0,\Lambda}(I) = \chi_I(-\Delta_\Lambda)$, où $I \subset \mathbb{R}$ contient 0, on a $\langle(\overline{V}_\Lambda - V_\Lambda)\varphi, \varphi\rangle \approx \|\nabla\varphi\|$. Donc, le fait de travailler à très basse énergie assure que le potentiel auxiliaire est une bonne approximation du potentiel original. De plus, par le terme à gauche de (1.2.26), le potentiel auxiliaire vérifie (1.2.17) sur le cube Λ, d'où on obtient l'estimée de Wegner.

Le même argument de moyennage produit une lacune spectrale en bas du spectre pour des versions à volume fini de H_ω, ce qui avec l'estimée de Combes–Thomas sert à montrer l'estimée d'échelle initiale, et à obtenir, donc, la localisation dans un intervalle d'énergie $[0, E_*]$. Ce raisonnement fut utilisé pour le modèle de Bernoulli dans [G08]. Ensuite, on utilise des critères issus de [GK03] pour savoir comment la longueur de l'intervalle de localisation E_* dépend explicitement des paramètres de l'ensemble de Delone sous-jacent au potentiel V_ω. Plus précisement :

Théorème 1.2.5. *Soit $M > 0$ fixé et D un ensemble de Delone de paramètres (r, R). Soit $H_\omega = -\Delta + V_\omega$ l'opérateur de Delone–Anderson associé sur $L^2(\mathbb{R}^d)$, où V_ω est donné par (1.2.14), avec des variables aléatoires appartenant à l'intervalle $[0, M]$, de manière que $\sigma(H_\omega) = [0, \infty)$ pour p.t. $\omega \in \Omega$. Il existe des constantes positives C' et A, qui dépendent des paramètres du modèle d, u, M et de la distribution de probabilité μ, et une énergie*

$$E_* = \frac{C'}{R^{2(d+1)}\,|\log AR|^{2/d}} \tag{1.2.27}$$

telles que pour tout intervalle $I \subset [0, E_*)$, on a $I \subset \Sigma_{MSA}$. En particulier, on a de la localisation dynamique dans $[0, E_*)$ pour la famille $\{H_\omega\}_{\omega\in\Omega}$.

La méthode de moyennage (1.2.26) est une approche élémentaire du problème d'obtenir (1.2.17). Son avantage, hors sa simplicité, est qu'elle peut s'utiliser dans le cadre continu ainsi comme dans le cadre discret, là où l'UCP n'est pas valable. Prenons, par exemple, le modèle d'Anderson

$$H_\omega = -\Delta + V_\omega \quad \text{ sur } l^2(\mathbb{Z}^d), \tag{1.2.28}$$

où

$$V_\omega(x) = \sum_{j\in(2\mathbb{Z})^d} \omega_j\delta_j(x), \tag{1.2.29}$$

où $\delta_j(\cdot)$ est le delta de Kronecker et ω_j sont des variables aléatoires indépendantes, identiquement distribuées supportées dans $[0, M]$, pour $M > 0$. Notons que V_ω est un potentiel invariant par rapport aux translations dans $(2\mathbb{Z})^d$, et donc (1.2.28) est un opérateur ergodique. Ceci implique que le spectre $\sigma(H_\omega) = [0, 2d+M]$ pour p.t. $\omega \in \Omega$. De plus, ce potentiel a des trous, i.e.,. $V_\omega = 0$ sur $\mathbb{Z}^d \setminus (2\mathbb{Z})^d$. En utilisant le moyennage spatiale (1.2.26), on peut suivre la même preuve de l'estimée de Wegner que dans le théorème précédent pour obtenir une estimée de Wegner pour (1.2.28) dans un intervalle $I = [0, E_d)$, où E_d est une constante qui dépend de la dimension d. Pour obtenir l'estimée de pas initial en bas du spectre, l'ergodicité du modèle nous permet de procéder de la manière standard dans le cadre ergodique, qui consiste à utiliser la densité d'états intégrée pour obtenir des asymptotiques de Lifshitz en bas du spectre. Puisque l'on peut vérifier les hypothèses de la MSA pour ce modèle ergodique, on obtient

Théorème 1.2.6. *Pour le modèle d'Anderson discret* (1.2.28), (1.2.29), *on a de la localisation dynamique en bas du spectre.*

Ce résultat est, à notre connaissance, le premier sur la localisation pour des potentiels ergodiques discrets avec des trous.

Nous prenons ensuite le cas où $H_0 = -\Delta + V_0$, avec V_0 un potentiel borné. En prenant des variables aléatoires appartenant à $[0, M]$, on a $E_0 = \inf \sigma(H_0) = \sigma(H_\omega)$ pour p.t. $\omega \in \Omega$. Si $V_0 \neq 0$, la méthode de moyennisation spatiale utilisée auparavant ne fonctionne plus en bas du spectre puisque $E_0 \neq 0$, et donc le potentiel auxiliaire n'est plus une bonne approximation du potentiel original. Si V_0 est périodique nous obtenons une estimée de Wegner optimale en dehors du spectre non perturbé en suivant le raisonnement de [CHK03, CHK07], sans utiliser le principe de continuation unique. Par ailleurs, dans le cas où H_0 a une densité d'états intégrée qui est Hölder continue, nous utilisons cette continuité pour remplacer le principe de continuation unique et obtenir ainsi une estimée de Wegner optimale. Sinon, pour le cas $H_0 = -\Delta + V_0$ où V_0 est simplement borné, on utilise les résultats sur des estimés de Wegner obtenus pour des potentiels de Delone dans [RMV12]. En ce qui concerne l'estimée d'échelle initiale, nous profitons du principe de continuation unique de [RMV12] pour obtenir la bonne décroissance de la résolvante par l'estimée de Combes–Thomas, sous une condition sur le désordre. Plus précisément, on suppose que la distribution de probabilité μ a un comportement assez plat sur les bords de son support. Ici on peut, comme auparavant, identifier la dépendance de l'énergie E_* des paramètres de l'ensemble de Delone et obtenir le théorème qui suit,

Théorème 1.2.7. *Soit D un ensemble de Delone de paramètres (r, R) et $H_\omega = -\Delta + V_0 + V_\omega$ l'opérateur de Delone–Anderson associé sur $L^2(\mathbb{R}^d)$, où V_0 est un potentiel borné et V_ω est donné par* (1.2.14). *On suppose qu'il existe des constantes c et $\tau \geq d/2$ telles que la distribution de probabilité μ supporté dans $[0, M]$ vérifie au bord*

$$\mu[0, t] \leq ct^\tau, \quad \textit{pour } t > 0 \textit{ petit.} \tag{1.2.30}$$

Soit $\beta \in (d/(2\tau), 1)$ fixé. Alors, il existe des constantes C' et C qui dépendent de d, M, I, u, V_0, β et une énergie E_* donnée par

$$E_* = E_0 + C' R^{-C\, R^{4/3} sgn(R-1)} \tag{1.2.31}$$

telles que pour tout intervalle ouvert $I \subset [E_0, E_*)$, assez petit, on a $I \subset \Sigma_{MSA}$. En particulier, H_ω a du spectre dynamiquement localisé dans $[E_0, E_*)$, où $E_0 = \inf \sigma(H_\omega)$ pour p.t. $\omega \in \Omega$.

Ensuite, nous nous intéressons au cas $H_\omega = -\Delta + V_0$, où V_0 est borné et V_ω est un potentiel de Delone–Bernoulli de la forme (1.2.14) où les variables aléatoires sont de Bernoulli, de paramètre β, tel que $0 \neq E_0 = \inf \sigma(H_0) = \inf \sigma(H_\omega)$ pour p.t. $\omega \in \Omega$. Sans perte de généralité, on suppose que $E_0 > 0$. Pour ceci, on utilise la MSA développée par [GK11], basée sur [BoK05]. Il suffit de montrer l'estimée d'échelle initiale, et on le fait en montrant qu'il existe, avec une assez bonne probabilité, une lacune spectrale au dessus de E_0 pour la version à volume fini de l'opérateur H_ω. Pour cela, on extrait de V_ω un potentiel auxiliaire V_K qui est du type Delone de paramètre maximal K_L, où K_L dépend de la taille de la boite Λ_L utilisée pour définir la version à volume fini de H_ω. Pour montrer que la perturbation V_K crée une lacune spectrale au dessus de l'infimum spectral E_0, on se sert d'une des conséquences du principe de continuation unique dans sa version quantitative, sur la perturbation de l'infimum spectrale (voir l'Appendice A.3 et

[RMV12, BdMLS11]). On utilise la constante du principe de continuation unique obtenue dans [RMV12], valable pour toute dimension. Cette constante dépend des paramètres de l'ensemble de Delone de caractéristique maximale R de la forme $C_{UCP} \approx R^{-R^{4/3}}$, qui est la même dépendance que dans le cas d'un réseau périodique, voir [BoK05, GK11]. Cette information, appliquée au potentiel V_K donne la taille de la lacune spectral au dessus de E_0, qui dans ce cas sera de l'ordre de $K_L^{-K_L^{4/3}}$. Ensuite, on utilise l'estimée de Combes–Thomas pour montrer la décroissance de la résolvante à l'intérieur de la lacune spectrale, qui prouve l'estimée d'échelle initiale, comme on l'a fait pour les variables aléatoires à densité régulière. Dans notre analyse, la forme explicite de la constante C_{UCP} joue un rôle central dans la preuve de l'existence d'une lacune spectrale (voir Appendice A.3).

Théorème 1.2.8. *Soit D un ensemble de Delone à paramètres (r, R) et $H_\omega = -\Delta + V_0 + V_\omega$ l'opérateur associé sur $L^2(\mathbb{R}^d)$, où V_ω est un potentiel de Delone–Bernoulli de paramètre β.*

i. En dimension $d \geq 2$, on a de la localisation dynamique en bas du spectre.

ii. En dimension $d = 1$, il existe une constante C, qui dépend de u et V_0, tel que si

$$\beta < e^{-CR}, \tag{1.2.32}$$

alors on a de la localisation dynamique en bas du spectre.

La restriction $d \geq 2$ est donnée par la décroissance que l'on obtient de l'estimée de Combes–Thomas. Or, en dimension $d = 1$, on se sert d'un autre principe de continuation unique obtenu pour des configurations d'impuretés périodiques dans [V96, KV02a]. Ces arguments se basent sur le lemme de Gromwall, et ne se servent pas de la périodicité du réseau. Dans l'Appendice A.2 nous suivons ces arguments pour des configurations de Delone et ainsi nous obtenons une constante qui dépend du paramètre maximal R de l'ensemble de Delone comme $C_{UCP} \approx R^{-1}e^{-R}$. Dans la preuve du Théorème 4.5.1, ceci donnera une lacune spectrale au-dessus de E_0 pour la perturbation V_K de taille $\approx K_L^{-1}e^{-K_L}$. Cette constante présente une amélioration par rapport à celle obtenue par [RMV12, BoK05, GK11]. Cependant, on a besoin de la condition (4.5.3) sur le désordre pour avoir la bonne décroissance de la résolvante d'après l'estimée de Combes–Thomas.

1.2.6 La densité d'états intégrée pour des opérateurs de Delone–Anderson

Comme nous expliquions au début de cette introduction, une conséquence majeure de l'ergodicité dans le modèle d'Anderson est l'existence de la limite de la fonction de comptage de valeurs propres normalisée, appelée aussi densité d'états intégrée à volume fini, défini par

$$\nu_{z,L}^{\omega}(E) = \frac{1}{L^d} \operatorname{tr} \chi_{(-\infty,E]}(H_\omega)\chi_{\Lambda_{z,L}}, \tag{1.2.33}$$

où $\chi_{\Lambda_{z,L}}$ est la fonction caractéristique du cube $\Lambda_{z,L}$ de centre $z \in \mathbb{R}^d$ et volume L^d. Cette définition est équivalente à (1.1.9). L'existence d'une telle limite se fonde sur le théorème ergodique de Birkhoff. Celui-ci permet de montrer que la suite $\nu_{z,L}^{\omega}(E)$, vue comme un processus stochastique ergodique, converge vers une quantité déterministe, obtenue par une moyenne sur l'espace de probabilités. Comme le modèle d'Anderson est invariant par rapport aux translations spatiales, le théorème ergodique nous permet de passer d'une moyenne spatiale à une moyenne en probabilité.

Dans le modèle d'Anderson avec des variables aléatoires à distributions décroissantes, qui n'est pas ergodique, on peut remplacer le théorème ergodique par la loi des grands nombres. Pour que cela fonctionne, on redéfinit la densité d'états à volume fini (1.2.33) et on remplace le facteur de normalisation L^d de manière convenable [BoeKS05, Boe03]. Ceci ressemble à l'approche utilisée dans des modèles à potentiels surfaciques, qui sont ergodiques dans une seule direction, où on étudie plutôt la notion de *densité d'états surfacique* [BoeKS05, BdMS03, BdMSS05]. Dans le cadre des opérateurs de Delone–Anderson, où ces arguments ne sont pas valables, on utilise le système dynamique de Delone coloré $\hat{X}_D$ (1.1.15) pour étendre la famille d'opérateurs H_ω à une famille $\{H_{P^\omega}\}_{P^\omega \in \hat{X}_D}$, de manière à retrouver de l'ergodicité. Dans ce cadre on utilise le théorème ergodique de Müller et Richard [MR12] qui permet de travailler avec des ensembles discrets colorés, où la couleur peut varier continûment, i.e., un espace de couleur $\mathbb{A} = I$, où I est un intervalle, est admissible. Nous montrons le théorème suivant,

Théorème 1.2.9. *Soient D un ensemble de Delone de paramètres (r, R) et $H_\omega = H_0 + V_\omega$ l'opérateur de Delone–Anderson associé agissant sur $L^2(\mathbb{R}^d)$. Si le système dynamique associé à D, X_D est uniquement ergodique, alors la limite* (1.2.33) *existe pour presque tout $\omega \in \Omega$. La limite est appelée densité d'états intégrée et on la dénote par ν.*

La preuve consiste à montrer la convergence vague des mesures associés à la fonction distribution (1.2.33). La mesure n dont ν est la fonction distribution, c'est-à-dire $\nu(E) = n((-\infty, E])$, est appelée la *mesure de densité d'états*. Le fait que l'ensemble de Delone est tel que le système dynamique associé soit uniquement ergodique est fondamental pour éliminer des ensembles négligeables pour lesquels le résultat du théorème ergodique de [MR12] ne se vérifie pas. Sans cette condition, le résultat d'existence dans le Théorème 1.2.9 serait pour presque tout élément de X_D, donc il pourrait arriver que l'ensemble qui nous interesse soit dans le complément, de mesure nulle.

Pour faire le lien entre la densité d'états intégrée et le spectre de H_ω on doit faire quelques suppositions sur la géometrie de l'ensemble de Delone. D'un côté, D doit vérifier la propriété de complexité locale finie : les motifs qui composent l'ensemble D doivent être, sauf translations, un nombre fini. Mais ce n'est pas suffisant, puisque tout motif de D doit être en plus répété un nombre infini de fois dans D. Ceci implique que la fréquence d'apparition de motifs doit être uniforme, mais en plus non nulle. On appelle cette propriété *fréquence d'apparition de motifs uniforme strictement positive*. De cette manière, toute région de D qui contribue à la présence d'une valeur propre à une échelle finie, est toujours présente dans la limite quand l'échelle tend vers l'infini.

Théorème 1.2.10. *Soit D un ensemble de Delone de paramètres (r, R) de complexité locale finie avec une fréquence d'apparition de motifs uniforme et strictement positive. Soit $H_\omega = H_0 + V_\omega$ l'opérateur de Delone–Anderson associé agissant sur $L^2(\mathbb{R}^d)$, avec une mesure de probabilité de simple site à support dans un alphabet fini. On a*

$$\operatorname{supp} n = \sigma(H_\omega), \qquad \textit{pour presque tout } \omega \in \Omega, \tag{1.2.34}$$

où $\operatorname{supp} n$ *est le support topologique de la mesure densité d'états* n . *Comme conséquence, il existe des ensembles $\Sigma_{pp}, \Sigma_{ac}, \Sigma_{sc} \subset \mathbb{R}$ tels que*

$$\sigma_\bullet(H_\omega) = \Sigma_\bullet, \quad \textit{pour } \bullet = pp, ac, sc. \tag{1.2.35}$$

Une fois que l'existence de la densité d'états est démontrée, on peut étudier son comportement en bas du spectre. Pour l'opérateur $H = -\Delta$, la loi asymptotique de Weyl donne le comportement [RSIV]

$$\nu(E) \sim C(E - E_0)^{d/2}, \tag{1.2.36}$$

pour des énergies E près de $E_0 = 0$. Plus généralement, pour un potentiel périodique $V_0 \neq 0$, la densité d'états intégrée de $H = -\Delta + V_0$ se comporte comme (1.2.36) pour d'énergies E près de $E_0 = \inf \sigma(H)$ [vH53].

Cependant, pour une perturbation de type d'Anderson, ce cadre change abruptement. Lifshitz [L65] observa que le comportement de la densité d'états intégrée est fortement affecté par la présence du désordre dans un milieu, et que en bas du spectre le comportement polynomial (1.2.36) devient exponentiel. Plus précisement, pour un opérateur $H_\omega = -\Delta + V_0 + V_\omega$ et pour des énergies très proches de $E_0 = \inf \sigma(H_0)$, on a

$$\nu(E) \sim c_1 e^{c_2 (E-E_0)^{-d/2}}, \tag{1.2.37}$$

pour des constantes positives c_1, c_2. On appelle ce dernier comportement, asymptotiques de Lifshitz [DV75, N77, P77, KM83, KS86]. Pour une perturbation du type Delone–Anderson du Laplacien, nous trouvons un résultat analogue au cas ergodique :

Théorème 1.2.11. *Soient D et $H_\omega = -\Delta + V_\omega$ vérifiant les hypothèses du Théorème* (1.2.9). *La densité d'états intégrée $\nu(E)$ a un comportement asymptotique de Lifshitz* (1.2.37) *en bas du spectre.*

La preuve suit les mêmes pas que dans le cadre ergodique. Cela consiste à obtenir des bornes supérieure et inférieure pour ν en utilisant l'encadrement de Dirichlet–Neumann. Pour obtenir cet encadrement, nous utilisons la sous-additivité et la super-additivité de processus $\operatorname{tr} \chi_{(-\infty,E]}(H_\omega|_{\Lambda_L})$ par rapport aux conditions au bord de Neumann et de Dirichlet, respectivement, et le fait que la mesure $\hat{\mu}$ sur l'espace coloré $\hat{X}_D$ est invariante par translation dans $\mathbb{R}^d$. Nous obtenons des bornes supérieure et inférieure qui dépendent des paramètres r et R de l'ensemble $P \in X_D$. Puisque ces paramètres sont les mêmes pour tous les éléments de X_D, les bornes obtenues sont uniformes sur X_D. On en déduit des bornes uniformes pour ν.

Rappelons que pour un potentiel V_0 périodique, le spectre de $H_0 = -\Delta + V_0$ est une réunion des bandes de spectre absolument continu, où on peut avoir des lacunes spectrales, i.e., des bandes disjointes. Prenons un potentiel d'Anderson V_ω, de taille $\|V_\omega\|_\infty$ assez petite, tel que le spectre de $H_\omega = H_0 + V_\omega$ présente des lacunes spectrales. On s'attend à que la densité d'états intégrée de H_ω vérifie (1.2.37) aux bords (internes) des bandes correspondants à ces lacunes spectrales. On appelle ce phénomène *asymptotiques de Lifshitz internes* [Mez87, S87, Kl99, KlW02]. En particulier, Klopp [Kl99] montra que si la densité d'états intégrée de l'opérateur H_0 se comporte comme (1.2.36) aux bords spectraux internes, alors la densité d'états intégrée de H_ω vérifie (1.2.37). Son analyse se base dans une approximation de $\nu(E)$ par les densités d'états associées à des opérateurs périodiques, pour lesquelles la convergence vers ν est exponentiellement rapide. Puisque l'on se sert de la théorie de Bloch-Floquet pour exploiter la périodicité des approximations, cette méthode ne marche pas dans le modèle de Delone–Anderson. L'étude des asymptotiques de Lifshitz internes dans le cadre non ergodique reste un problème ouvert.

L'importance de l'étude des asymptotiques de Lifshitz est lié à la présence de la localisation [Kl95, Kl02a, KSS98]. Dans ce cadre, on peut utiliser le Théorème 1.2.11 pour obtenir l'estimée d'échelle initiale (1.2.3), en remplaçant l'analyse fait par le moyennage spatiale dans la preuve du Théorème 1.2.5. On obtient de la localisation dynamique en bas du spectre pour une perturbation de type Delone–Anderson du Laplacien, et la même description de l'intervalle de localisation obtenue auparavant (1.2.27). Cependant, la preuve du Théorème 1.2.7 en bas du spectre en utilisant des asymptotiques de Lifshitz reste hors de notre portée.

Pour illustrer comment la géométrie de l'ensemble de Delone D a des conséquences sur l'existence de la densité d'états intégrée, nous étudions un opérateur de Delone qui ne vérifie les hypothèses des théorèmes précédents.

Considérons une suite de cubes centrés à l'origine, $\{\Lambda_{L_k}\}_{k\in\mathbb{N}}$, avec $L_{k+1} = L_k^{\alpha}$, $\alpha > 1$. Soit $\mathbb{N}_e = \{2k : k \in \mathbb{N}\}$, $\mathbb{N}_o = \{2k-1 : k \in \mathbb{N}\}$. On décompose $\mathbb{R}^d$ de la manière suivante,

$$\mathbb{R}^d = \bigcup_{k=1}^{\infty} A_k, \quad A_k = \overline{\Lambda}_{L_k} \setminus \Lambda_{L_{k-1}}, \, \Lambda_{L_0} = \emptyset. \tag{1.2.38}$$

Soit D l'ensemble de Delone défini par

$$D = \left(\bigcup_{k\in\mathbb{N}_e} q_1\mathbb{Z}^d \cap A_k \right) \cup \left(\bigcup_{k\in\mathbb{N}_o} q_2\mathbb{Z}^d \cap A_k \right), \tag{1.2.39}$$

et H_D l'opérateur de Delone associé, agissant sur $\mathrm{L}^2(\mathbb{R}^d)$, donné par

$$H_D = H_0 + \sum_{\gamma\in D} u(x-\gamma). \tag{1.2.40}$$

Proposition 1.2.3. *Soit $\nu_{H_D,L}$ la densité d'états intégrée à volume fini de H_D. La limite de $\nu_{H_D,L}$ quand L tend vers l'infini n'existe pas.*

Ceci est à la base une conséquence du fait que l'ensemble D ne vérifie pas la propriété de fréquence d'apparition de motifs uniforme strictement positive. On considère les suites $\{A_k\}_{k\in\mathbb{N}_e}$ et $\{A_k\}_{k\in\mathbb{N}_o}$ séparément : la restriction de H à la suite $\{A_k\}_{k\in\mathbb{N}_e}$ est équivalente à la restriction d'un opérateur périodique H_{q_1} de période q_1, tandis que la restriction de H à la suite $\{A_k\}_{k\in\mathbb{N}_o}$ se comporte comme un opérateur périodique H_{q_2} de période q_2. Donc, la densité d'états intégrée de H se comportera de manière alternée comme celle de H_{q_1} et de H_{q_2}. En prenant la limite quand k tend vers l'infini, on obtiendra des résultats différents pour $k \in \mathbb{N}_e$ et pour $k \in \mathbb{N}_o$.

Chapter 2

Characterization of the Anderson-Metal Insulator Transition for non ergodic operators

Contents

2.1 The model and main results

In this chapter, we study the characterization of the Anderson metal–insulator transition for non ergodic random Schrödinger operators in both annealed and quenched regimes, based on a dynamical approach of localization. We extend known results for ergodic operators into this more general setting and obtain annealed and quenched versions of the transport transition. The characterization of the Anderson metal-insulator transition *à la* Germinet-Klein consists in proving the equivalence of the region where we can apply the Bootstrap MSA, i.e., where a Wegner estimate and an initial length scale estimate hold, and the region where the operator H_ω exhibits dynamical localization. Therefore, the main ingredient in our proof is a generalization of the Bootstrap Multiscale Analysis of Germinet and Klein to fit the non ergodic setting. In this section we recall the results that compose the transport characterization and dedicate the following two sections of the chapter to their proof. These results are contained in the article "Characterization of the Characterization of the Anderson Metal–Insulator Transition for Non Ergodic Operators and Application" published in Annales Henri Poincaré [RM12]. To complete our study, we comment on the case where only weaker versions of the Wegner estimate hold. In the last section we prove that in this situation the characterization fails to hold for dimension $d \geq 2$.

For the reader's convenience, and to make this chapter self-contained, we recall the notation and definitions stated in the Introduction. For $x \in \mathbb{R}^d$ we denote by $\|x\|$ the usual Euclidean norm while the supremum norm is defined as $|x|_\infty = \max_{1\leq i\leq d} |x_i|$, where $|\cdot|$ stands for absolute value.

Given $x \in \mathbb{R}^d$ and $L > 0$ we denote by either $B(x, L)$ or $B_L(x)$, the ball of center x and radius L in the $\|\cdot\|$-norm, while the set

$$\Lambda_L(x) = \left\{ y \in \mathbb{R}^d : |y - x|_\infty < \frac{L}{2} \right\}$$

defines the cube of side L centered at x, also denoted as $\Lambda_{x,L}$. We denote the volume of a Borel set $\Lambda \subset \mathbb{R}^d$ with respect to the Lebesgue measure as $|\Lambda| = \int_{\mathbb{R}^d} \chi_\Lambda(x) d^d x$, where χ_Λ is the characteristic function of the set Λ. We will often write $\chi_{x,L}$ for $\chi_{\Lambda_L(x)}$ and denote by $\|f\|_{x,L}$ or $\|f\|_{\Lambda_L(x)}$ the norm of f in $\mathrm{L}^2(\Lambda_{x,L})$.

We denote by $\mathcal{C}_c^\infty(\Lambda)$ the vector space of real-valued infinitely differentiable functions with compact support contained in Λ, with $C_{c,+}^\infty(\Lambda)$ being the subclass of nonnegative functions.

We denote by $\mathcal{B}(\mathcal{H})$ the Banach space of bounded linear operators on the Hilbert space $\mathcal{H}$. For a closed, densely defined operator A with adjoint A^*, we denote its domain by $\mathcal{D}(A) \subset \mathrm{L}^2(\Lambda)$ and by $\|A\| = \sup\{\|A\phi\|; \quad \|\phi\|_2 = 1\}$ its (uniform) norm if bounded. We define its absolute value by $|A| = \sqrt{A^* A}$ and, for $p > 1$, we define its (Schatten) p-norm in the Banach space $\mathcal{J}_p(\mathrm{L}^2(\Lambda))$ as $\|A\|_p = (\mathrm{tr}\, |A|^p)^{1/p}$. In particular, $\mathcal{J}_1$ is the space of trace-class operators and $\mathcal{J}_2$, the space of Hilbert-Schmidt operators. We write $\langle x \rangle = \sqrt{(1 + \|x\|^2)}$ and use $\langle X \rangle$ to denote the operator given by multiplication by the function $\langle x \rangle$.

For convenience we denote a constant C depending only on the parameters a,b,... by $C_{a,b,\ldots}$.

Through this chapter, we assume H_ω is a general random operator, not necessarily ergodic (see 1.1.3), satisfying properties

(R) $V_\omega = V_\omega^+ + V_\omega^-$, where V_ω^+ and V_ω^- are real valued measurable processes on Ω such that for $\mathbb{P}-$a.e. $\omega : 0 \leq V_\omega^+ \in \mathrm{L}^1_{loc}(\mathbb{R}^d)$ and V_ω^- is relatively form-bounded with respect to $-\Delta$, with relative bound < 1, i.e. there are nonnegative constants $\Theta_1 < 1$ and Θ_2 independent of ω such that for all $\psi \in \mathcal{D}(\nabla)$ we have

$$|\langle \psi, V_\omega^- \psi \rangle| \leq \Theta_1 \|\nabla\psi\|^2 + \Theta_2 \|\psi\|^2 \text{ for } \mathbb{P}\text{-a.e. } \omega.$$

(IAD) There exists $\varrho > 0$ such that for any bounded sets $B_1, B_2 \subset \mathbb{R}^d$ with dist$(B_1, B_2) > \varrho$, the processes $\{V_\omega(x) : x \in B_1\}$ and $\{V_\omega(x) : x \in B_2\}$ are independent.

In the case $H_0 = H_B$, the unperturbed Landau Hamiltonian on $\mathrm{L}^2(\mathbb{R}^2)$

$$H_B = (-i\nabla - \mathbf{A})^2 \quad \text{with } \mathbf{A} = \frac{B}{2}(x_2, -x_1), \tag{2.1.1}$$

where $\mathbf{A}$ is the vector potential and B is the strength of the magnetic field, we ask $\mathbf{A}(x) \in \mathrm{L}^2_{\mathrm{loc}}(\mathbb{R}^2; \mathbb{R}^2)$ to satisfy the diamagnetic inequality so we can obtain trace estimates for the Landau Hamiltonian from those of the Laplacian.

It follows that H_ω is a semibounded selfadjoint operator for $\mathbb{P}$-a.e. ω. Moreover, the mapping $\omega \to H_\omega$ is measurable for $\mathbb{P}$-a.e. ω, we denote its spectrum by σ_ω.

For the following assumption we need the notion of a *finite volume operator*, the restriction of H_ω to either an open box $\Lambda_L(x)$ with Dirichlet boundary condition or to the closed box $\overline{\Lambda}_L(x)$ with periodic boundary conditions. In this way, we obtain a well defined random operator $H_{\omega,x,L}$ acting on $\mathrm{L}^2(\Lambda_L(x))$ defined by

$$H_{\omega,x,L} = H_{0,x,L} + \lambda V_{\omega,x,L}.$$

We denote its spectrum by $\sigma_{\omega,x,L}$ and by $R_{\omega,x,L}(z) = (H_{\omega,x,L} - z)^{-1}$ its resolvent operator. We define the spectral projections $P_\omega(J) = \chi_J(H_\omega)$ and $P_{\omega,x,L}(J) = \chi_J(H_{\omega,x,L})$ for $J \subset \mathbb{R}$ a Borel set. When stressing the dependence on λ, it will be added to the subscript.

Definition 2.1.1.

(UWE) We say that H_ω satisfies a uniform Wegner estimate with Hölder exponent s in an open interval $\mathcal{J}$, i.e., for every $E \in \mathcal{J}$ there exists a constant Q_W, bounded on compact subintervals of $\mathcal{J}$ and $0 < s \leq 1$ such that

$$\sup_{x\in\mathbb{R}^d} \mathbb{E}\{\operatorname{tr} P_{\omega,x,L}(E-\eta, E+\eta)\} \leq Q_W \eta^s L^d, \tag{2.1.2}$$

for all $\eta > 0$ and $L \in 2\mathbb{N}$. It satisfies a uniform Wegner estimate at an energy E if it satisfies a uniform Wegner estimate in an open interval $\mathcal{J}$ such that $E \in \mathcal{J}$.

To describe the dynamics, we consider the random moment of order $p \geq 0$ at time t for the time evolution in the Hilbert-Schmidt norm, initially spatially localized in a square of side one around $u \in \mathbb{Z}^2$ and localized in energy by the function $\mathcal{X} \in C^\infty_{c,+}(\mathbb{R})$, i.e.,

$$M_{u,\omega}(p, \mathcal{X}, t) = \| \langle X - u\rangle^{p/2} e^{-itH_\omega} \mathcal{X}(H_\omega)\chi_u\|_2^2. \tag{2.1.3}$$

We next consider its time average,

$$\mathcal{M}_{u,\omega}(p, \mathcal{X}, T) = \frac{2}{T}\int_0^\infty e^{-2t/T} M_{u,\omega}(p, \mathcal{X}, t)dt. \tag{2.1.4}$$

Definition 2.1.2.

1. We say that H_ω exhibits strong Hilbert-Schmidt (HS-) dynamical localization in the open interval I if for all $\mathcal{X} \in \mathcal{C}^\infty_{c,+}(I)$ we have

$$\sup_{u\in\mathbb{Z}^2} \mathbb{E}\{\sup_{t\in\mathbb{R}} M_{u,\omega}(p, \mathcal{X}, t)\} < \infty \quad \text{for all } p \geq 0.$$

   We say that H_ω exhibits strong Hilbert-Schmidt (HS-) dynamical localization at an energy E if there exists an open interval I with $E \in I$, such that there is strong HS-dynamical localization in the open interval.

2. The strong insulator region for H_ω is defined as

$$\Sigma_{SI} = \{E \in \mathbb{R} : H_\omega \text{ exhibits strong HS-dynamical localization at } E\}.$$

   Note that if there exists a $\delta > 0$ such that $\operatorname{dist}(E, \sigma_\omega) > \delta$ for almost every ω, then $E \in \Sigma_{SI}$.

Given $\theta > 0$, $E \in \mathbb{R}$, $x \in \mathbb{Z}^d$ and $L \in 6\mathbb{N}$, we say that the box $\Lambda_L(x)$ is (θ, E)-*suitable* for H_ω if $E \notin \sigma_{\omega,x,L}$ and

$$\|\Gamma_{x,L} R_{\omega,x,L}(E) \chi_{x,L/3}\|_{x,L} \leq \frac{1}{L^\theta},$$

where $\Gamma_{x,L} = \chi_{\bar{\Lambda}_{L-1}(x) \setminus \Lambda_{L-3}(x)}$. If we replace the polynomial decay $1/L^\theta$ by $e^{-mL/2}$ we say that the box $\Lambda_L(x)$ is (m, E)-*regular* for H_ω.

As a first step towards the transport characterization in non ergodic models, we extend the Bootstrap MSA [GK01] to the non ergodic setting:

Theorem 2.1.3. *Let H_ω be a random Schrödinger operator satisfying a uniform Wegner estimate in an open interval $\mathcal{J}$ with Hölder exponent s and assumptions (R), (IAD). Given $\theta > d$, for each $E \in \mathcal{J}$ there exists a finite scale $\mathcal{L}_\theta(E) = \mathcal{L}(\theta, E, Q_W, d, s)$, bounded in compact subintervals of $\mathcal{J}$, such that if for $\mathcal{L} > \mathcal{L}_\theta(E)$ the following holds*

$$\inf_{x \in \mathbb{Z}^d} \mathbb{P}\{\Lambda_{\mathcal{L}}(x) \text{ is } (\theta, E)\text{-suitable}\} > 1 - \frac{1}{841^d}, \tag{2.1.5}$$

then there exists $\delta_0 > 0$ and $C_\zeta > 0$ such that

$$\sup_{u \in \mathbb{Z}^d} \mathbb{E}\left(\sup_{\|f\| \leq 1} \|\chi_{x+u} f(H_\omega) P_\omega(I(\delta_0)) \chi_u\|_2^2 \right) \leq C_\zeta e^{-|x|^\zeta}, \tag{2.1.6}$$

for $0 < \zeta < 1$, where $I(\delta_0) = [E - \delta_0, E + \delta_0]$. Moreover, $E \in \Sigma_{SI}$ and we have the following properties,

(SUDEC) Summable uniform decay of eigenfunction correlations: for a.e. $\omega \in \Omega$, the Hamiltonian H_ω has pure point spectrum in $I \subset \Sigma_{SI}$ with finite multiplicity. Let $\{\epsilon_{n,\omega}\}_{n \in \mathbb{N}}$ be an enumeration of the distinct eigenvalues of H_ω in I. Then for each $\zeta \in]0, 1[$ and $\epsilon > 0$ we have, for every $x, u \in \mathbb{Z}^d$,

$$\|\chi_{x+u}\phi\| \|\chi_u \varphi\| \leq C_{I,\zeta,\epsilon,\omega} \|T_u^{-1}\phi\| \|T_u^{-1}\varphi\| \langle x+u \rangle^{\frac{d+\epsilon}{2}} \langle u \rangle^{\frac{d+\epsilon}{2}} e^{-|x|^\zeta}, \tag{2.1.7}$$

for all $\phi, \varphi \in$ Ran $P_\omega(\{\epsilon_{n,\omega}\})$.

(SULE) Semi-uniformly localized eigenfunctions: Let $I \subset \Sigma_{SI}$ and let $\{\epsilon_{n,\omega}\}_{n \in \mathbb{N}}$ be the enumeration of the distinct eigenvalues of H_ω in I. For every $\epsilon > 0$, there exists a constant $m_\epsilon > 0$ and, for almost every $\omega \in \Omega$, there exists a constant $C_{\epsilon,\omega} < \infty$, such that if $(\varphi_{n,\omega})_{n \in \mathbb{N}}$ are the normalized eigenfunctions associated to the eigenvalues $\{\epsilon_{n,\omega}\}_{n \in \mathbb{N}}$ in I, there exists $\{x_{n,\omega}\}_{n \in \mathbb{N}}$, such that for every $n \in \mathbb{N}$ and $x \in \mathbb{Z}^d$, we have

$$\|\chi_x \varphi_{n,\omega}\| \leq C_{\epsilon,\omega} e^{m_\epsilon (\log|x_{n,\omega}|)^{1+\epsilon}} e^{-m_\epsilon |x - x_{n,\omega}|}. \tag{2.1.8}$$

Moreover, the localization centers $x_{n,\omega}$ can be ordered in such a way that

$$|x_{n,\omega}| \geq \tilde{C}_\omega n^{1/(4\nu)}, \tag{2.1.9}$$

for some finite constant $\tilde{C}_\omega > 0$, and $\nu > d/4$ as in property UGEE, see (2.2.8).

(DFP) Decay of the Fermi projections: for $E \in \Sigma_{SI}$ and for any $\zeta \in]0,1[$ we have

$$\sup_{u \in \mathbb{Z}^d} \mathbb{E}\{\|\chi_{x+u} P_\omega((-\infty, E])\chi_u\|_2^2\} \leq C_{\zeta,\lambda,E} e^{-|x|^\zeta}, \tag{2.1.10}$$

where the constant $C_{\zeta,E}$ is locally bounded in E.

Remark 2.1.4. *The condition* (2.1.5) *is called the* initial length scale estimate *(ILSE) of the Bootstrap MSA. In practice is often useful to prove the equivalent estimate [GK04, Theorem 4.2]: For some $\theta > d$, we have*

$$\limsup_{L\to\infty} \inf_{x \in \mathbb{Z}^d} \mathbb{P}\{\Lambda_{\mathcal{L}}(x) \text{ is } (\theta, E)\text{-suitable}\} = 1. \tag{2.1.11}$$

Definition 2.1.5. The multiscale analysis region for H_ω is defined as the set of energies where we can perform the bootstrap MSA, i.e.

$$\begin{aligned}\Sigma_{MSA} = &\{E \in \mathbb{R} : H_\omega \text{ satisfies a uniform Wegner estimate at } E \text{ and} \\ &\text{(ILSE) holds for some } \mathcal{L} > \mathcal{L}_\theta(E)\}.\end{aligned}$$

By Theorem 2.1.3, we have $\Sigma_{MSA} \subset \Sigma_{SI}$.

We introduce the (lower) transport exponent in the annealed regime:

$$\beta(p, \mathcal{X}) = \liminf_{T\to\infty} \frac{\log_+ \sup_u \mathbb{E}(\mathcal{M}_{u,\omega}(p, \mathcal{X}, T))}{p \log T}, \tag{2.1.12}$$

for $p \geq 0$, $\mathcal{X} \in C^\infty_{c,+}(\mathbb{R})$, where $\log_+ t = \max\{0, \log t\}$, and define the p-th local transport exponent at the energy E, by

$$\beta(p, E) = \inf_{I \ni E} \sup_{\mathcal{X} \in C^\infty_{c,+}(I)} \beta(p, \mathcal{X}), \tag{2.1.13}$$

where I denotes an open interval. The exponents $\beta(p, E)$ provide a measure of the rate of transport in wave packets with spectral support near E. Since they are increasing in p, we define the local (lower) transport exponent $\beta(E)$ by

$$\beta(E) = \lim_{p\to\infty} \beta(p, E) = \sup_{p>0} \beta(p, E). \tag{2.1.14}$$

With the help of this transport rate we can define two complementary sets in the energy axis for fixed $B > 0$, $\lambda > 0$, the region of *dynamical localization*

$$\Xi^{DL} = \{E \in \mathbb{R} : \; \beta(E) = 0\}, \tag{2.1.15}$$

also called the trivial transport region (TT) in [GK04] and the region of *dynamical delocalization*

$$\Xi^{DD} = \{E \in \mathbb{R} : \; \beta(E) > 0\}, \tag{2.1.16}$$

also called the *weak metallic transport region* (WMT), in [GK04]. Note that $\Sigma_{SI} = \Xi^{DL}$.

Recall that by Theorem 2.1.3, $\Sigma_{MSA} \subset \Sigma_{SI}$. Now, to show that these two sets are equivalents, as in the ergodic setting, we aim to use the information on the dynamics of the operator, through the transport exponent β, plus the Wegner estimate, to obtain the uniform initial length scale estimate (2.1.5). The following result is an improvement of [GK04, Theorem 2.11] for the non ergodic setting,

Theorem 2.1.6. *Let H_ω be a Schrödinger operator satisfying a uniform Wegner estimate with Hölder exponent s in an open interval $\mathcal{J}$ and assumptions (R), (IAD). Let $\mathcal{X} \in C^\infty_{c,+}(\mathbb{R})$ with $\mathcal{X} \equiv 1$ on some open interval $J \subset \mathcal{J}$, $\alpha \geq 0$ and $p > p(\alpha, s) := 12\frac{d}{s} + 2\alpha\frac{d}{s}$. If*

$$\liminf_{T\to\infty} \sup_{u\in\mathbb{Z}^d} \frac{1}{T^\alpha}\mathbb{E}\left(\mathcal{M}_{u,\omega}(p, \mathcal{X}, T)\right) < \infty, \tag{2.1.17}$$

then $J \subset \Sigma_{MSA}$. In particular, it follows that (2.1.17) *holds for any $p \geq 0$.*

Moreover, we can extend this result to a *quenched* regime. Under the assumption that the set of realizations for which the growth rate of the random moment as a function of T is faster than polynomial has an asymptotically small probability, we can obtain the initial step to start the Bootstrap MSA:

Theorem 2.1.7. *Let H_ω be a Schrödinger operator satisfying a uniform Wegner estimate with Hölder exponent s in an open interval $\mathcal{J}$ and assumptions (R), (IAD). Let $\mathcal{X} \in C^\infty_{c,+}(\mathbb{R})$ with $\mathcal{X} \equiv 1$ on some open interval $J \subset \mathcal{J}$, $\alpha \geq 0$ and $p > p(\alpha, s) := 15\frac{d}{s} + 2\alpha\frac{d}{s}$. If*

$$\liminf_{T\to\infty} \sup_{u\in\mathbb{Z}^d} T^{\frac{s}{d}}\mathbb{P}(\mathcal{M}_{u,\omega}(p, \mathcal{X}, T) > T^\alpha) = 0, \tag{2.1.18}$$

then $J \subset \Sigma_{MSA}$. In particular, it follows that (2.1.18) *holds for any $p \geq 0$.*

If $\alpha > s/d$, then $\Sigma_{DL} \subset \Sigma_{MSA}$, and we retrieve a transport behavior analogous to that of the annealed regime.

Next, we consider the case where only weak versions of the Wegner estimate hold (see [DS01, vDK89, BoK05]):

Definition 2.1.8. We say the random operator H_ω satisfies a (polynomially) weak Wegner estimate in an open interval I with parameters $\eta \in (0,1)$ and $q > 0$, if for every energy $E \in I$ and there exists a length scale $L_0 \in 6\mathbb{N}$, such that for $L > L_0$

$$\mathbb{P}\left(\mathrm{dist}(E, \sigma_{\omega,x,L}) \leq e^{-L^\eta}\right) \leq L^{-q}, \tag{2.1.19}$$

uniformly with respect to x.

In the case of an Anderson potential, where the single-site probability distribution is μ, we define the the global modulus of continuity of as $s(\mu, \epsilon) = \sup_{E\in\mathbb{R}} \mu([E-\epsilon, E+\epsilon])$. We say μ is *log-Hölder continuous* of parameter $\nu > 0$ if for some $c_\mu > 0$ and $\epsilon \in (0,1)$ we have

$$s(\mu, \epsilon) \leq \frac{c_\mu}{|\log\epsilon|^\nu}.$$

The following result links this kind of measures with optimal and weak Wegner estimates [GHV08, Lemma 13]

Lemma 2.1.9 ([GHV08]). *Let H_ω be a random Schrödinger operator and $I_0 \subset \mathbb{R}$ a bounded interval. Assume that there exists constants Q_W, L_0 such that for every $\epsilon > 0$ and $L \geq L_0$ an* optimal *Wegner estimate holds*

$$\mathbb{E}(\operatorname{tr} P_{\omega,L}([E-\epsilon, E+\epsilon])) \leq Q_W s(\mu, \epsilon). \tag{2.1.20}$$

If the measure μ is log-Hölder continuous with parameter $\nu > \frac{q+d}{\eta}$, then H_ω satisfies a weak Wegner estimate with parameters $\eta, q > 0$ in I_0 for all $L \geq L_1$, where $L_1 = \max\{L_0, (Q_W c_\mu)^{1/\delta}\}$ and $\delta := \nu\eta - q - d > 0$.

To describe the dynamics of the system, for $\sigma > 0$, $\zeta \in (0,1)$ we define the (σ,ζ)-subexponential random moment at time t for the time evolution in the Hilbert-Schmidt norm, initially spatially localized in a cube of side one around $u \in \mathbb{Z}^d$ and localized in energy by a function $\mathcal{X} \in C^\infty_{c,+}(\mathbb{R})$

$$M_{u,\omega}(\sigma,\zeta,\mathcal{X},t) = \| e^{\frac{\sigma}{2}|X-u|^\zeta} e^{-itH_\omega} \mathcal{X}(H_\omega)\chi_u\|_2^2. \tag{2.1.21}$$

Define its time average as

$$\mathcal{M}_{u,\omega}(\sigma,\zeta,\mathcal{X},T) = \frac{2}{T}\int_0^\infty e^{-2t/T} M_{u,\omega}(\sigma,\zeta,\mathcal{X},t)dt. \tag{2.1.22}$$

Definition 2.1.10. Analogously to the definition of the strong insulator region, define the region of *sub-exponential* dynamical localization as

$$\Sigma_{SEDL} = \left\{ E \in I : \sup_{u\in\mathbb{Z}^2} \mathbb{E}\left(\sup_{t\in\mathbb{R}} M_{u,\omega}(\sigma,\zeta,\mathcal{X},t)\right) < \infty \text{ for some } \sigma > 0, \zeta \in (0,1) \right\}. \tag{2.1.23}$$

We adapt Theorem 2.1.6 (see also [GK04, Theorem 2.11]) to the weak Wegner estimate as follows:

Theorem 2.1.11. *Let H_ω be a Schrödinger operator as defined above, satisfying a weak Wegner estimate in an open interval $\mathcal{J}$ with parameters $\eta \in (0,1)$, and $q > 0$. Let $\mathcal{X} \in C^\infty_{c,+}(\mathbb{R})$ with $\mathcal{X} \equiv 1$ on some open interval $J \subset \mathcal{J}$, $\alpha \geq 0$, and take $\zeta > \eta$ and $\sigma > 0$. If*

$$\liminf_{T\to\infty} \sup_{u\in\mathbb{Z}^d} \frac{1}{T^\alpha} \mathbb{E}\left(\mathcal{M}_{u,\omega}(\sigma,\zeta,\mathcal{X},T)\right) < \infty, \tag{2.1.24}$$

then $J \subset \Sigma_{MSA}$. In particular, $\Sigma_{SEDL} \subset \Sigma_{MSA}$.

However, the transport characterization does not hold under these assumptions, since the MSA with a polynomially weak Wegner estimate yields polynomial, yet not sub-exponential, dynamical localization.

In dimension $d = 1$, for singular measures whose support is not concentrated in a single point, for example Bernoulli measures, one can prove the following (exponentially) weak form of the Wegner estimate [CKM87, KLS90, DSS02] : for every $\eta \in (0,1)$, $\sigma > 0$, there exists $L_0 \in \mathbb{N}$ and $\alpha > 0$ such that

$$\mathbb{P}\left(\operatorname{dist}(E, \sigma_{\omega,x,L}) \leq e^{-\sigma L^\eta}\right) \leq e^{-\alpha L^\eta} \tag{2.1.25}$$

for every $E \in I$ and $L \geq L_0$.

The Bootstrap MSA can still be performed with this Wegner estimate, yielding sub-exponential decay of the operator kernel and in the same lines of proof of the previous theorem one can prove the equivalence $\Sigma_{MSA} = \Sigma_{SEDL}$.

2.2 The Multiscale Analysis for the non-ergodic setting: Proof of Theorem 2.1.3

We recall some properties the operator $H_{\omega,x,L}$ must satisfy, among others, to perform the Bootstrap MSA [GK01, GK04]:

(SLI) *The Simon–Lieb inequality*: For any compact interval $I \subset \mathcal{C}I$, there exists a finite constant γ_I such that, given lengthscales $L, l, l' \in 2\mathbb{N}$, and points $x, y, z \in \mathbb{Z}^d$ with

$$\Lambda_{l'}(y) \subset \Lambda_{l-3}(z) \subset \Lambda_{L-3}(x), \tag{2.2.1}$$

then, for a.e. $\omega \in \Omega$, of $E \in I$ and $E \notin \sigma(H_{\omega,x,L}) \cup \sigma(H_{\omega,z,l})$, we have

$$\|\Gamma_{x,L} R_{\omega,x,L}(E)\, \chi_{y,l'}\| \leq \gamma_I \|\Gamma_{z,l} R_{\omega,z,l}(E)\, \chi_{y,l'}\|\, \|\Gamma_{x,L} R_{\omega,x,L}(E)\, \Gamma_{z,l}\|. \tag{2.2.2}$$

(EDI) *The eigenfunction decay inequality*: for any compact interval $I \subset \mathcal{C}I$, there exists a finite constant $\tilde{\gamma}_I$ such that for a.e. $\omega \in \Omega$, given a generalized eigenfunction φ of H_ω with generalized eigenvalue $E \in J$, for any $x \in \mathbb{Z}^d$ and $L \in 2\mathbb{N}$ with $E \notin \sigma(H_{\omega,x,L})$, we have

$$\|\chi_x\, \varphi\| \leq \tilde{\gamma}_I \|\Gamma_{x,L} R_{\omega,x,L}(E) \chi_x\|\, \|\Gamma_{x,L} \varphi\|. \tag{2.2.3}$$

A consequence that will be of use is that for $y \in \operatorname{supp} \Gamma_{x,L}$,

$$\|\chi_x \varphi\| \leq d \tilde{\gamma}_I L^{d-1} \|\Gamma_{x,L} R_{\omega,x,L}(E) \chi_x\| \|\chi_y \varphi\|. \tag{2.2.4}$$

(NE) For any compact interval $I \subset \mathcal{C}I$ there exists a finite constant C_I such that, for all $x \in \mathbb{Z}^d$ and $L \in 2\mathbb{N}$,

$$\mathbb{E}\left\{\operatorname{tr} P_{\omega,x,L}(I)\right\} \leq C_I L^d. \tag{2.2.5}$$

2.2.1 Generalized eigenfunction expansion

In order to proof Theorem 2.1.3 we have to construct a generalized eigenfunction expansion adapted to the non ergodic case. Compared to [GK01, Section 2.3] we shall use a family of weighted spaces rather than just one in particular, using translations in $u \in \mathbb{Z}^2$ of the operator T defined there and thus without using translation invariance in the proofs.

Let T_u be the operator in $\mathcal{H}$ given by multiplication by the function $(1 + |x-u|^2)^\nu$, where $\nu > d/4$, $u \in \mathbb{Z}^2$. We define the weighted spaces $\mathcal{H}^u_\pm$ as

$$\mathcal{H}^u_\pm = L^2(\mathbb{R}^d, (1+|x-u|^2)^{\pm 2\nu} dx; \mathbb{C}). \tag{2.2.6}$$

The sesquilinear form

$$\langle \phi_1, \phi_2 \rangle_{\mathcal{H}^u_+, \mathcal{H}^u_-} = \int \overline{\phi}_1 \phi_2(x) dx \quad \text{ for } \phi_1 \in \mathcal{H}^u_+ \text{ , } \phi_2 \in \mathcal{H}^u_-$$

makes $\mathcal{H}^u_+$ and $\mathcal{H}^u_-$ conjugates dual to each other and we denote by † the conjugation with respect to this duality. The natural injections $\iota^u_+ : \mathcal{H}^u_+ \to \mathcal{H}$ and $\iota^u_- : \mathcal{H} \to \mathcal{H}^u_-$ are continuous with dense range, with $(\iota_+^u)^\dagger = \iota_-^u$. The operators $T_{u,+} : \mathcal{H}_+^u \to \mathcal{H}$ and $T_{u,-} : \mathcal{H} \to \mathcal{H}_-^u$ defined by $T_{u,+} = T_u \iota_+^u$, $T_{u,-} = \iota_-^u T_u$ on $\mathcal{D}(T_u)$ are unitary with $T_{u,-} = T_{u,+}^\dagger$. Note that

$$\|\chi_{x,L}\|_{\mathcal{H},\mathcal{H}_+^u} = \|\chi_{x,L}\|_{\mathcal{H}_-^u,\mathcal{H}} \leq C_{L,d,\nu}(1+|x-u|^2)^\nu, \tag{2.2.7}$$

for all $x \in \mathbb{R}^d$ and $L > 0$.

With this redefinition we can follow [GK01], restating assumption GEE for non ergodic operators. We consider a fixed open interval $\mathcal{I}$ and we recall that $P_\omega(J) = \chi_J(H_\omega)$ is the spectral projection of the operator H_ω on a Borel set $J \subset \mathbb{R}$.

(UGEE) *For some $\nu > d/4$, the set $\mathcal{D}_+^{u,\omega} = \{\phi \in \mathcal{D}(H_\omega) \cap \mathcal{H}_+^u : \ H_\omega\phi \in \mathcal{H}_+^u\}$ is dense in $\mathcal{H}_+$ and an operator core for H_ω for $\mathbb{P}-$a.e. ω and all u. There exists a bounded function f, strictly positive on the spectrum of H_ω such that, }*

$$\sup_u \ \mathrm{tr}_{\mathcal{H}} \left(T_u^{-1} f(H_\omega) P_\omega(\mathcal{I}) T_u^{-1}\right) < \infty, \tag{2.2.8}$$

for $\mathbb{P}-$a.e. ω.

If UGEE holds, for almost every ω and all u we have

$$\mathrm{tr}_{\mathcal{H}} \left(T_u^{-1} P_\omega(J \cap \mathcal{I}) T_u^{-1}\right) < \infty, \tag{2.2.9}$$

for all bounded sets J. Thus with probability one, for all u

$$\mu_{u,\omega}(J) = \mathrm{tr}_{\mathcal{H}} \left(T_u^{-1} P_\omega(J \cap \mathcal{I}) T_u^{-1}\right) \tag{2.2.10}$$

is a spectral measure for the restriction of H_ω to the Hilbert space $P_\omega(\mathcal{I})\mathcal{H}$, and for every bounded set J,

$$\mu_{u,\omega}(J) < \infty. \tag{2.2.11}$$

Then, we have a generalized eigenfunction expansion as in [GK01, Section 2]: for every u, there exists a $\mu_{u,\omega}$-locally integrable function $\mathbf{P}_{u,\omega}(\tilde{\lambda})$ from $\mathbb{R}$ into $\mathcal{T}_1(\mathcal{H}_+^u, \mathcal{H}_-^u)$, the space of trace class operators from $\mathcal{H}_+^u$ to $\mathcal{H}_-^u$, with

$$\mathbf{P}_{u,\omega}(\tilde{\lambda}) = \mathbf{P}_{u,\omega}(\tilde{\lambda})^\dagger \tag{2.2.12}$$

and

$$\mathrm{tr}_{\mathcal{H}} \left(T_{u,-}^{-1} \mathbf{P}_{u,\omega}(\tilde{\lambda}) T_{u,+}^{-1}\right) = 1 \quad \text{for } \mu_{u,\omega}\text{-a.e. } \tilde{\lambda}, \tag{2.2.13}$$

such that

$$\iota_-^u P_\omega(J \cap \mathcal{I}) \iota_+^u = \int_J \mathbf{P}_{u,\omega}(\tilde{\lambda}) d\mu_{u,\omega}(\tilde{\lambda}) \quad \text{for bounded Borel sets } J, \tag{2.2.14}$$

where the integral is the Bochner integral of $\mathcal{T}_1(\mathcal{H}_+^u, \mathcal{H}_-^u)$-valued functions.

The following (a restatement of assumption SGEE), is a stronger version of UGEE:

(USGEE) *We have that UGEE holds with*

$$\sup_u \mathbb{E}\left([\operatorname{tr}_{\mathcal{H}}\left(T_u^{-1}f(H_\omega)P_\omega(\mathcal{I})T_u^{-1}\right)]^2\right) < \infty. \tag{2.2.15}$$

So for every bounded set J,

$$\sup_u \mathbb{E}(\mu_{u,\omega}(J)^2) < \infty. \tag{2.2.16}$$

2.2.2 Kernel Decay and Dynamical Localization

Following the arguments in [GK01] for ergodic operators, we can show that HS-strong dynamical localization is a consequence of the applicability of the Bootstrap MSA for the non ergodic setting ([GK01, Theorem 3.4] with the stronger initial ILSE (2.1.5) instead of the original one).

We can restate Lemma 2.5 and Lemma 4.1 [GK01] as follows, extending the proofs to our new definitions,

Lemma 2.2.1. *Let H_ω be a random operator satisfying assumption GEE. We have with probability one, for all u, that for $\mu_{u,\omega}$-almost every $\tilde{\lambda}$,*

$$\|\chi_x \mathbf{P}_{u,\omega}(\tilde{\lambda})\chi_y\|_1 \le C(1+|x-u|^2)^\nu(1+|y-u|^2)^\nu, \tag{2.2.17}$$

for all $x, y \in \mathbb{R}^d$, with C a finite constant independent of $\tilde{\lambda}, \omega$ and u.

Suppose, moreover, that assumption EDI in [GK01] is satisfied in some compact interval $I_0 \subset \mathcal{I}$. Given $I \subset I_0$, $m > 0$, $L \in 6\mathbb{N}$ and $x, y \in \mathbb{Z}^d$, if $\omega \in R(m, L, I, x, y)$, with $R(m, L, I, x, y)$ defined as in (2.2.23)*, then*

$$\|\chi_x \mathbf{P}_{u,\omega}(\tilde{\lambda})\chi_y\|_2 \le Ce^{-mL/4}(1+|x-u|^2)^\nu(1+|y-u|^2)^\nu, \tag{2.2.18}$$

for $\mu_{u,\omega}$-almost all $\tilde{\lambda} \in I$, with $C = C(m, d, \nu, \tilde{\gamma}_{I_0})$, where $\tilde{\gamma}_{I_0}$ is the constant on assumption EDI.

Proof of Theorem 2.1.3. To apply the MSA in the non ergodic case we first need to verify for an operator satisfying only properties R, IAD and UWE, the standard assumptions SLI, EDI [GK01], plus NE and USGEE, which are stronger assumptions than those stated in the mentioned article.

As for SLI and EDI, these are deterministic assumptions that hold for each $\omega \in \Omega$ and their proof, done in [GK04, Appendix A], relies on property R, with no use of ergodicity. In the same appendix we see that assumption NE is uniform on cubes centered in $x \in \mathbb{R}^d$ and relies on property R so it holds in our more general setting. The same is true for [GK04, Lemma A.3], and can be extended in an analog way to the case $H_0 = H_B$ [BGKS05, Section 2.1], proving the first part of USGEE (and UGEE).

As for the trace estimate (2.2.15) , for the case $H_0 = -\Delta$ it follows from [GK04, Lemma A.4] and [KKS02, Theorem 1.1], taking $V = \langle X - u\rangle^{-2\nu}$ there, the result being uniform in u. It can be extended to the case $H_0 = H_B$ as in [BGKS05, Proposition 2.1].

To obtain the basic result of MSA [GK01, Theorem 3.4] we need conditions IAD, SLI, NE and UWE to follow an analog iteration procedure. Recall that in their article, Germinet and Klein take two versions of MSA by Figotin and Klein, improve their estimates yielding other two MSA and then bootstrapping them to obtain the strongest result out of the weakest hypothesis, so in order to extend this results to the non ergodic setting we reformulate this methods. Each step consists of a purely geometric deterministic part where we use SLI, and therefore it does not depend on the placement of the boxes were we perform the procedure, and a probabilistic part, where we use UWE instead of WE to obtain an estimate on the probability of having bad events, in a stronger sense than the usual, that is, uniform with respect to the placement of the box in space.

We begin with the single energy multiscale analyses, Theorems 5.1 and 5.6 [GK01], which in our non-ergodic setting consists in estimating the decay of

$$p_L = \sup_{x\in\mathbb{Z}^d} p_{x,L}, \tag{2.2.19}$$

where

$$p_{x,L} = \mathbb{P}\{\Lambda_L(x) \text{ is bad}\} \tag{2.2.20}$$

(here a box is bad if it is not (θ, E)-*suitable* for H_ω). In the ergodic case we need only to consider $p_{0,L}$. Hypothesis (2.1.5) ensures we can follow the same iteration procedure in all boxes centered in $x \in \mathbb{Z}^d$, where $p_{0,L}$ is thus replaced by p_L. We use properties SLI and UWE instead of WE, and the deterministic arguments remain the same, since they do not depend on the location of the box. Considering a Hölder exponent s in WE implies that the choice of the initial length scale will also depend on s.

Next we consider the energy interval multiscale analyses, Theorems 5.2 and 5.7 [GK01], which in our general setting consists in estimating

$$\tilde{p}_L = \sup_{\substack{x,y\in\mathbb{Z}^d \\ |x-y|>L+\varrho}} \tilde{p}_{x,y,L}, \tag{2.2.21}$$

with

$$\tilde{p}_{x,y,L} = \mathbb{P}\{R(m, L, I(\delta_0), x, y)^c\}, \tag{2.2.22}$$

where $I(\delta_0) = [E - \delta_0, E + \delta_0]$, for some $\delta_0 > 0$ and

$$R(m, L, I(\delta_0), x, y) = \{\omega : \text{ for every } E \in I(\delta_0),\ \Lambda_L(x) \text{ or } \Lambda_L(y) \text{ is good}\} \tag{2.2.23}$$

(here a box is good if it is (m, E)-*regular* for H_ω, with m to be specified later). In the ergodic case it suffices to consider $\tilde{p}_{x,y,L}$. We can thus follow the original iteration procedure on this estimate, replacing $\tilde{p}_{x,y,L}$ by $\tilde{p}_L$, obtaining an analog of [GK01, Eq. 3.4], i.e., there exists $\delta_0 > 0$ such that given any ζ, $0 < \zeta < 1$ there is a length scale $L_0 < \infty$ and a mass $m_\zeta = m(\zeta, L_0) > 0$ such that if we set $L_{k+1} = [L_k^\alpha]_{6\mathbb{N}}$, $0 < \alpha < \zeta^{-1}$,$k = 0, 1, 2, ...$ we have

$$\inf_{\substack{x,y\in\mathbb{Z}^d\\|x-y|>L+\varrho}} \mathbb{P}\{R(m_\zeta, L_k, I(\delta_0), x, y)\} \geq 1 - e^{-L_k^\zeta}. \tag{2.2.24}$$

To derive results on the spectrum and the dynamics of the operator from this estimate we need to consider also conditions EDI and USGEE. Thus, with Lemma 2.2.1 in hand, (2.2.24) and USGEE we can follow the proof of [GK01, Theorem 3.8] with minor modifications. We want to show that if (2.2.24) holds we have that for any $0 < \zeta < 1$, there is a finite constant C_ζ such that

$$\sup_u \mathbb{E}\left(\sup_{\|f\|\leq 1} \|\chi_{x+u} f(H_\omega) P_\omega(I(\delta_0))\chi_u\|_2^2\right) \leq C_\zeta e^{-|x|^\zeta}, \tag{2.2.25}$$

For this, we consider the pair of points x, y as the pair $x+u, u$, and fix $x \in \mathbb{Z}^d$ and k such that $L_{k+1} + \varrho > |x| > L_k + \varrho$. We split the expectation in (2.2.25) in two parts: the first one over the set $R(m_\zeta, L_k, I(\delta_0), x+u, u)$ and the second one over its complement, which has probability less than $e^{-L_k^\zeta}$, uniformly in u, by (2.2.24). We follow the arguments in [GK01, Eq. 4.8-4.13]. By (2.2.14) and Lemma 2.2.1 we can write, for a positive constant C_1,

$$\sup_{\|f\|\leq 1} \|\chi_{x+u} f(H_\omega) P_\omega(I(\delta_0))\chi_u\|_2 \leq C_1 e^{-L_k^\zeta} \mu_{u,\omega}(I). \tag{2.2.26}$$

This implies,

$$\begin{aligned} &\sup_u \mathbb{E}\left(\sup_{\|f\|\leq 1} \|\chi_{x+u} f(H_\omega) P_\omega(I(\delta_0))\chi_u\|_2^2 ; R(m_\zeta, L_k, I(\delta_0), x+u, u)\right) \\ &\leq C_1^2 \sup_u \mathbb{E}\{(\mu_{u,\omega}(I(\delta_0)))^2\} e^{-2L_k^\zeta}. \end{aligned} \tag{2.2.27}$$

As for the expectation over $R(m_\zeta, L_k, I(\delta_0), x+u, u)^c$, (2.2.24) implies that

$$\sup_u \mathbb{P}(R(m_\zeta, L_k, I(\delta_0), x+u, u)^c) < e^{-L_k^\zeta},$$

this yields,

$$\begin{aligned} &\sup_u \mathbb{E}\left(\sup_{\|f\|\leq 1} \|\chi_{x+u} f(H_\omega) P_\omega(I(\delta_0))\chi_u\|_2^2 ; R(m_\zeta, L_k, I(\delta_0), x+u, u)^c\right) \\ &\leq 4^\nu \sup_u \mathbb{E}\{(\mu_{u,\omega}(I(\delta_0)))^2\}^{\frac{1}{2}} e^{-\frac{1}{2}L_k^\zeta}, \end{aligned} \tag{2.2.28}$$

where we use the fact that by (2.2.10) we can write

$$\|\chi_{x+u} f(H_\omega) P_\omega(I(\delta_0))\chi_u\|_2^2 \leq \|f\|^2 \|P_\omega(I(\delta_0))\chi_u\|_2^2 \leq C\|f\| \mu_{u,\omega}(I(\delta_0)). \tag{2.2.29}$$

Combining (2.2.27) and (2.2.28), using USGEE we obtain the desired decay, namely (2.2.25).

Now we can prove a strong version of dynamical localization as in [GK01, Corollary 3.10]. Notice that, if $p > 2$

$$\langle X-u\rangle^p = \sum_{x\in\mathbb{Z}^d}(1+\|y-u\|^2)^{p/2}\chi_x(y) \le C_d \sum_{x\in\mathbb{Z}^d}(1+\|x-u\|^2)^{p/2}\chi_x(y)$$
$$= C_d \sum_{x\in\mathbb{Z}^d}(1+\|x\|^2)^{p/2}\chi_{x+u}(y), \tag{2.2.30}$$

so we have,

$$\begin{aligned}
&\|\langle X-u\rangle^{p/2} f(H_\omega)P_\omega(I(\delta_0))\chi_u\|_2^2 \\
&\quad = \operatorname{tr}\left[\chi_u f(H_\omega)P_\omega(I(\delta_0))\langle X-u\rangle^p P_\omega(I(\delta_0))f(H_\omega)\chi_u\right] \\
&\quad \le C_d \sum_{x\in\mathbb{Z}^d}(1+\|x\|^2)^{p/2}\operatorname{tr}\left[\chi_u f(H_\omega)P_\omega(I(\delta_0))\chi_{x+u}P_\omega(I(\delta_0))f(H_\omega)\chi_u\right] \\
&\quad = C_d \sum_{x\in\mathbb{Z}^d}(1+\|x\|^2)^{p/2}\|\chi_{x+u}f(H_\omega)P_\omega(I(\delta_0))\chi_u\|_2^2.
\end{aligned}\tag{2.2.31}$$

Taking the expectation and then the supremum over $u \in \mathbb{Z}^2$, by (2.2.25) we obtain strong HS-dynamical localization in the energy interval $I(\delta_0)$.

Following the proof of [GK06, Corollary 3], after adapting [GK06, Theorem 1] to our setting we obtain the summable uniform decay of eigenfunction correlations SUDEC. As for property DFP, it is a consequence of (2.2.25) combined with [BGK04, Theorem 1.4], which is a deterministic result also valid in our setting, in the lines of [GK06, Theorem 3]. The proof of SULE follows in the same lines as [GK01, Theorem 3.11], as a consequence of (2.2.24) and UGEE.

□

2.3 The weak metallic transport region: Proofs of Theorems 2.1.6 and 2.1.7

Here we can proceed as in [GK04]. First we state the following Lemma, which is an intermediate result in the proof of [GK04, Lemma 6.4], adapted to the UWE with Hölder exponent s. We consider a cube $\Lambda_L(x)$ with arbitrary x so we omit it from the notation.

Lemma 2.3.1. *Let H_ω be a random Schrödinger operator satisfying a uniform Wegner estimate in an open interval $\mathcal{I}$, with Wegner constant Q_W and Hölder exponent s. Let $p_0 > 0$ and $\gamma > d$. For each $E \in \mathcal{I}$, there exists $\mathcal{L} = \mathcal{L}(d, E, Q_W, \gamma, p_0, s)$ bounded on compact subsets of $\mathcal{I}$, such that, given $L \in 2\mathbb{N}$ with $L \ge \mathcal{L}$, and subsets B_1 and B_2 of Λ_L(not necessarily disjoint) with $B_1 \subset \Lambda_{L-5/2}$ and $\bar{\Lambda}_{L-1} \setminus \Lambda_{L-3} \subset B_2$, then for each $a > 0$ and $0 < \epsilon \le 1$ we have*

$$\mathbb{P}\left(\|\chi_2 R_{\omega,L}(E+i\epsilon)\chi_1\|_L > \frac{a}{4}\right) \le \mathbb{P}\left(\|\chi_2 R_\omega(E+i\epsilon)\chi_1\| > \frac{a}{L^\gamma}\right) + \frac{p_0}{10}, \tag{2.3.1}$$

and

$$\mathbb{P}\left(\|\chi_2 R_{\omega,L}(E)\chi_1\|_L > \frac{a}{2}\right) \leq \mathbb{P}\left(\|\chi_2 R_\omega(E+i\epsilon)\chi_1\| > \frac{a}{L^\gamma}\right) + Q_W\left(\frac{4\epsilon}{a}\right)^{s/2} L^d + \frac{p_0}{10}, \tag{2.3.2}$$

where χ_i stands for χ_{B_i}, $i = 1, 2$.

Proof of Theorem 2.1.6. By the same arguments used in [GK04, Theorem 4.2], it suffices to show that, under condition (2.1.18), for each $E \in J$ there is some $\theta > d/s$ such that

$$\limsup_{L\to\infty} \inf_{y\in\mathbb{Z}^d} \mathbb{P}\left(\|\Gamma_{y,L} R_{\omega,y,L}(E)\chi_{y,L/3}\|_{y,L} \leq \frac{1}{L^\theta}\right) = 1, \tag{2.3.3}$$

i.e. the starting condition for the bootstrap MSA, (2.1.5), in its strong version, holds at some finite scale $L > \mathcal{L}_\theta(E)$.

Let $E \in J$, $\theta > d/s$ and $L \in 6\mathbb{N}$. We start by estimating

$$P_{E,L} := \sup_y \mathbb{P}\left(\|\Gamma_{y,L} R_{\omega,y,L}(E)\chi_{y,L/3}\|_{y,L} > \frac{1}{L^\theta}\right). \tag{2.3.4}$$

We decompose as in [GK04, Eq. 6.26-6.28], using

$$\chi_{y,L} = \chi_{y,2L/3} + \chi_{y,L\setminus 2L/3}, \quad \text{where } \chi_{y,L\setminus 2L/3} = \chi_{y,\Lambda_L\setminus\Lambda_{2L/3}},$$

so (for simplicity we omit the subscript y from the norm)

$$P_{E,L} \leq \sup_y \mathbb{P}\left(\frac{1}{4L^\theta} < \|\Gamma_{y,L} R_{\omega,L}(E+i\epsilon)\chi_{y,L/3}\|_L\right) \tag{2.3.5}$$

$$+ \sup_y \mathbb{P}\left(\frac{1}{2L^\theta} < \epsilon\|R_{\omega,L}(E+i\epsilon)\|_L \|\Gamma_{y,L} R_{\omega,L}(E)\chi_{y,2L/3}\|_L\right) \tag{2.3.6}$$

$$+ \sup_y \mathbb{P}\left(\frac{1}{4L^\theta} < \epsilon\|R_{\omega,L}(E)\|_L \|\chi_{y,L\setminus 2L/3} R_{\omega,L}(E+i\epsilon)\chi_{y,L/3}\|_L\right). \tag{2.3.7}$$

To estimate the first term we use (2.3.1) with $a = L^{-\theta}$. As for the rest, we use (2.3.2) and (2.3.1), respectively, with $a = 1$, plus the uniform Wegner estimate. We obtain

$$P_{E,L} \leq \sup_y \mathbb{P}\left(\frac{1}{L^{\theta+\gamma}} < \|\Gamma_{y,L} R_\omega(E+i\epsilon)\chi_{y,L/3}\|\right) \tag{2.3.8}$$

$$+ \sup_y \mathbb{P}\left(\frac{1}{L^\gamma} < \|\Gamma_{y,L} R_\omega(E+i\epsilon)\chi_{y,2L/3}\|\right) \tag{2.3.9}$$

$$+ \sup_y \mathbb{P}\left(\frac{1}{L^\gamma} < \|\chi_{y,L\setminus 2L/3} R_\omega(E+i\epsilon)\chi_{y,L/3}\|\right) \tag{2.3.10}$$

$$+ Q_I(4\epsilon)^{s/2} L^d + 2Q_I \epsilon^s L^{\theta s+d} + \frac{3p_0}{10}, \tag{2.3.11}$$

for $L > \mathcal{L}$, with $\mathcal{L}$ as in Lemma 2.3.1, where $\gamma > d/s$, $0 < \epsilon \leq 1$, $0 < p_0 < 1$ and $Q_I = \sup_{E \in I} Q_W < \infty$ (recall the Wegner constant Q_W in (2.1.2) depends on E). Set

$$L = L(I,\epsilon) := \left[\left(\frac{p_0}{20 Q_I \epsilon^s} \right)^{1/(\theta s + d)} \right]_{6\mathbb{N}}, \tag{2.3.12}$$

so that

$$Q_I (4\epsilon)^{s/2} L^d \leq \frac{p_0}{10} \quad \text{and} \quad 2 Q_I \epsilon^s L^{\theta s + d} \leq \frac{p_0}{10}.$$

We first estimate,

$$\sup_y \mathbb{P}\left(\frac{1}{L^{\theta+\gamma}} < \|\Gamma_{y,L} R_\omega(E + i\epsilon) \chi_{y,L/3}\| \right). \tag{2.3.13}$$

To do this, we decompose the norm using the function $\mathcal{X}(H_\omega)$ that localizes in energy , yielding

$$\sup_y \mathbb{P}\left(\frac{1}{2L^{\theta+\gamma}} < \|\Gamma_{y,L} R_\omega(E + i\epsilon) \mathcal{X}(H_\omega) \chi_{y,L/3}\| \right) \tag{2.3.14}$$

$$+ \sup_y \mathbb{P}\left(\frac{1}{2L^{\theta+\gamma}} < \|\Gamma_{y,L} R_\omega(E + i\epsilon)(1 - \mathcal{X}(H_\omega)) \chi_{y,L/3}\| \right). \tag{2.3.15}$$

For the second term we use Chebyshev's inequality and follow [GK04, Eq. 6.32 - 6.34], so we can bound it by $p_0/12$.

Estimating in the same way the terms (2.3.9) and (2.3.10) we obtain that for L big enough,

$$P_{E,L} \leq \sup_y \mathbb{P}\left(\frac{1}{2L^{\theta+\gamma}} < \|\Gamma_{y,L} R_\omega(E + i\epsilon) \mathcal{X}(H_\omega) \chi_{y,L/3}\| \right) \tag{2.3.16}$$

$$+ \sup_y \mathbb{P}\left(\frac{1}{2L^{\gamma}} < \|\Gamma_{y,L} R_\omega(E + i\epsilon) \mathcal{X}(H_\omega) \chi_{y,2L/3}\| \right) \tag{2.3.17}$$

$$+ \sup_y \mathbb{P}\left(\frac{1}{2L^{\gamma}} < \|\chi_{y,L \backslash 2L/3} R_\omega(E + i\epsilon) \mathcal{X}(H_\omega) \chi_{y,L/3}\| \right) + \frac{3p_0}{4}. \tag{2.3.18}$$

As for the first term,

$$\mathbb{P}\left(\frac{1}{2L^{\theta+\gamma}} < \|\Gamma_{y,L} R_\omega(E + i\epsilon) \mathcal{X}(H_\omega) \chi_{y,L/3}\| \right)$$

$$\leq 2L^{\theta+\gamma} \mathbb{E}\left(\|\Gamma_{y,L} R_\omega(E + i\epsilon) \mathcal{X}(H_\omega) \chi_{y,L/3}\| \right) \tag{2.3.19}$$

$$\leq 2L^{\theta+\gamma} \sum_{u \in \tilde{\Lambda}_{L/3}(y)} \mathbb{E}\left(\|\Gamma_{y,L} R_\omega(E + i\epsilon) \mathcal{X}(H_\omega) \chi_u\| \right). \tag{2.3.20}$$

For any u fixed, given a compact subinterval $I \subset J$ and $M > 0$ we set :

$$A_{u,M,I,\epsilon} = \left\{ E \in I : \ \mathbb{E}\left(\|\langle X - u \rangle^{p/2} R_\omega(E + i\epsilon) \mathcal{X}(H_\omega) \chi_u\|_2^2 \right) \leq M \epsilon^{-(\alpha+1)} \right\}.$$

We have, taking $T=\epsilon^{-1}$ and using [GK04, Lemma 6.3]

$$\begin{aligned}
|I\setminus A_{u,M,I,\epsilon}| &\leq \frac{1}{M\epsilon^{-(\alpha+1)}}\int_{\mathbb{R}}\mathbb{E}\left(\|\langle X-u\rangle^{p/2}R_\omega(E+i\epsilon)\mathcal{X}(H_\omega)\chi_u\|_2^2\right)dE \\
&= \frac{2\pi}{MT^{\alpha+1}}\int_0^\infty e^{-2t/T}\mathbb{E}\left(\|\langle X-u\rangle^{p/2}e^{-itH_\omega}\mathcal{X}(H_\omega)\chi_u\|_2^2\right)dt \\
&\leq \frac{\pi}{MT^\alpha}\sup_u \mathbb{E}\left(\mathcal{M}_{u,\omega}(p,\mathcal{X},T)\right).
\end{aligned} \tag{2.3.21}$$

Remark 2.3.2. *Notice that the analogous sets $A_{k,I,M}$ in the proof [GK04, Theorem 2.11] do not work in the non ergodic setting, so we need to consider a family of sets $A_{u,M,I,\epsilon}$, indexed by u.*

By hypothesis (2.1.17) we can pick a sequence $T_k\to\infty$ such that for k big enough, we have $\sup_u \mathbb{E}\left(\mathcal{M}_{u,\omega}(p,\mathcal{X},T_k)\right) < CT_k^\alpha$, then for the corresponding sequence $\epsilon_k\to 0^+$ we have

$$|I\setminus A_{u,M,I,\epsilon_k}|\leq \frac{C}{M}. \tag{2.3.22}$$

Notice that this bound is uniform in u.

Thus, for an $E\in I$ fixed and $\epsilon_k=T_k^{-1}$, either $E\in A_{u,I,M,\epsilon_k}$ in which case we have,

$$\begin{aligned}
&\mathbb{E}\left(\|\Gamma_{y,L_k}R_\omega(E+i\epsilon_k)\mathcal{X}(H_\omega)\chi_u\|\right) \\
&\qquad\leq C_{p,d}L_k^{-p/2}\mathbb{E}\left(\|\langle X-u\rangle^{p/2}R_\omega(E+i\epsilon_k)\mathcal{X}(H_\omega)\chi_u\|_2\right) \\
&\qquad\leq C_{p,d}L_k^{-p/2}\mathbb{E}\left(\|\langle X-u\rangle^{p/2}R_\omega(E+i\epsilon_k)\mathcal{X}(H_\omega)\chi_u\|_2^2\right)^{1/2} \\
&\qquad\leq C_{p,d}L_k^{-p/2}M^{1/2}\epsilon_k^{-(\alpha+1)/2},
\end{aligned} \tag{2.3.23}$$

where we write $L_k=L(I,\epsilon_k)$, or else, $E\in I\setminus A_{u,M,I,\epsilon_k}$, so by (2.3.22) there exists $E_u\in A_{u,I,M,\epsilon_k}$ such that

$$|E-E_u|\leq\frac{C}{M}$$

and so, by the resolvent identity and the definition of $A_{u,M,I,\epsilon}$,

$$\begin{aligned}
\mathbb{E}\left(\|\Gamma_{y,L_k}R_\omega(E+i\epsilon_k)\mathcal{X}(H_\omega)\chi_u\|\right) &\leq \mathbb{E}\left(\|\Gamma_{y,L_k}R_\omega(E_u+i\epsilon_k)\mathcal{X}(H_\omega)\chi_u\|\right) \\
&\quad+|E-E_u|\mathbb{E}\left(\|R_\omega(E+i\epsilon_k)\|\|R_\omega(E_u+i\epsilon_k)\|\right) \\
&\leq C_{p,d}L_k^{-p/2}M^{1/2}\epsilon_k^{-(\alpha+1)/2}+\frac{C}{M\epsilon_k^2}.
\end{aligned} \tag{2.3.24}$$

Therefore,

$$\mathbb{P}\left(\frac{1}{2L_k^{\theta+\gamma}} < \|\Gamma_{y,L_k} R_\omega(E+i\epsilon_k)\mathcal{X}(H_\omega)\chi_{y,L_k/3}\|\right) \tag{2.3.25}$$

$$\leq C'_{p,d} L_k^{\theta+\gamma-p/2+d} M^{1/2} \epsilon_k^{-(\alpha+1)} + C''_{p,d} \frac{L_k^{\theta+\gamma+d}}{M\epsilon_k^2}. \tag{2.3.26}$$

The remaining terms (2.3.17) and (2.3.18) are estimated in the same way, using the fact that $\operatorname{dist}(\bar{\Lambda}_{L-1} \setminus \Lambda_{L-3}, \Lambda_{\frac{2L}{3}}) \geq \frac{L}{3} - \frac{3}{2}$ and $\operatorname{dist}(\Lambda_{L \setminus \frac{2L}{3}}, \Lambda_{\frac{2L}{3}}) \geq \frac{L}{6}$. For these terms we obtain an estimate as (2.3.25) with constants $C'^{(2)}_{p,d}, C''^{(2)}_{p,d}$ and $C'^{(3)}_{p,d}$, $C''^{(3)}_{p,d}$, respectively, and with no θ in the exponent of L. Denote by $C_{p,d}$ the maximal constant, and since $L^\theta < L^{\theta+\gamma}$, the estimate on (2.3.25) using $C_{p,d}$ will imply the same estimate on (2.3.17) and on (2.3.18).

Now, for p such that $p > p'(\alpha, s) = \alpha\frac{2d}{s} + 12\frac{d}{s}$, we can find $\theta, \gamma > d/s$ for which

$$p > 5\theta + 3\gamma + 2d + (\alpha+1)(\theta s + d)/s, \tag{2.3.27}$$

so if we set

$$M = L_k^{3\theta+\gamma}, \tag{2.3.28}$$

and recall

$$\epsilon_k^{-(\alpha+1)/2} = C_{p_0,Q_I} L_k^{(\alpha+1)(\theta s+d)/2s}, \quad \epsilon_k^{-2} = C'_{p_0,Q_I} L_k^{-2(\theta s+d)/s}. \tag{2.3.29}$$

we obtain, for k big enough depending on $d, I, p, \alpha, \theta, \gamma, s, p_0, Q_I$,

$$C'_{p,d} L_k^{\theta+\gamma-p/2+d} M^{1/2} \epsilon_k^{-(\alpha+1)} < p_0/24 \tag{2.3.30}$$

and

$$C''_{p,d} \frac{L_k^{\theta+\gamma+d}}{M\epsilon_k^2} < p_0/24, \tag{2.3.31}$$

so there exists a sequence $L_k \to \infty$ such that for k big enough,

$$\mathbb{P}\left(\frac{1}{2L_k^{\theta+\gamma}} < \|\Gamma_{y,L_k} R_\omega(E+i\epsilon_k)\mathcal{X}(H_\omega)\chi_{y,L_k/3}\|\right) < \frac{p_0}{12}. \tag{2.3.32}$$

The same argument shows that the terms (2.3.17) and (2.3.18) are smaller than $p_0/12$, for k big enough.

Inserting this in (2.3.16)-(2.3.18) we see that

$$\limsup_{k\to\infty} \sup_y \mathbb{P}\left(\frac{1}{L_k^\theta} < \|\Gamma_{y,L_k} R_{\omega,y,L_k}(E)\chi_{y,L_k/3}\|_{L_k}\right) \leq p_0, \tag{2.3.33}$$

Since $0 < p_0 < 1$ is arbitrary, we conclude that (2.3.3) holds for each $E \in I$.

$\square$

Proof of Theorem 2.1.7. From equation (2.3.3) to equation (2.3.18) the previous proof remains valid in the current setting. We will only estimate (2.3.16), since the remaining terms (2.3.17) and (2.3.18) can be estimated in the same way. Notice that

$$\mathbb{P}\left(\frac{1}{2L^{\theta+\gamma}} < \|\Gamma_{y,L} R_\omega(E+i\epsilon)\mathcal{X}(H_\omega)\chi_{y,L/3}\|\right)$$

$$\leq \mathbb{P}\left(\frac{1}{2L^{\theta+\gamma}} < \sum_{u\in\tilde{\Lambda}_{L/3}(y)} \|\Gamma_{y,L} R_\omega(E+i\epsilon)\mathcal{X}(H_\omega)\chi_u\|\right)$$
$$\leq \sum_{u\in\tilde{\Lambda}_{L/3}(y)} \mathbb{P}\left(\frac{1}{2L^{\theta+\gamma+d}} < \|\Gamma_{y,L} R_\omega(E+i\epsilon)\mathcal{X}(H_\omega)\chi_u\|\right). \tag{2.3.34}$$

To estimate the r.h.s of the last inequality, the following following lemma is crucial,

Lemma 2.3.3. *There exists $\mathcal{L} = \mathcal{L}(I, p, \theta, \gamma, d, \alpha, s, p_0, Q_I)$ such that for any $u \in \tilde{\Lambda}_{L/3}(y)$ with $L = L(I, \epsilon)$ as in* (2.3.12)*, $L \geq \mathcal{L}$ and $E \in I$ fixed, if*

$$p > p(\theta, \gamma, d, \alpha, s) := \alpha\frac{(\theta s + d)}{s} + 9\theta + 3\gamma + 2d + \frac{d}{s}, \tag{2.3.35}$$

then, for $T = \epsilon^{-1}$,

$$\left\{\omega :\ \|\Gamma_{y,L} R_\omega(E+i\epsilon)\mathcal{X}(H_\omega)\chi_u\| > \frac{1}{2L^{\theta+\gamma+d}}\right\} \subset \{\omega :\ \mathcal{M}_{u,\omega}(p, \mathcal{X}, T) > T^\alpha\}. \tag{2.3.36}$$

Now, if $p > p(\alpha, s) := 15\frac{d}{s} + 2\alpha\frac{d}{s}$, then there exist $\theta, \gamma > d/s$ such that $p > p(\theta, \gamma, d, \alpha, s) > p(\alpha, s)$ so Lemma 2.3.3 holds yielding, for $L = L(I, \epsilon)$ as in (2.3.12) big enough,

$$\mathbb{P}\left(\frac{1}{2L^{\theta+\gamma}} < \|\Gamma_{y,L} R_\omega(E+i\epsilon)\mathcal{X}(H_\omega)\chi_{y,L/3}\|\right)$$
$$\leq C_{p_0,Q_I} T^{\frac{s}{2}} \sup_u \mathbb{P}(\mathcal{M}_{u,\omega}(p, \mathcal{X}, T) > T^\alpha), \tag{2.3.37}$$

where C_{p_0,Q_I} comes from $L^d = C_{p_0,Q_I} T^{\frac{s}{2}}$, by (2.3.12).

By hypothesis (2.1.18), we can pick a sequence $T_k \to \infty$ such that for k big enough

$$T_k^{\frac{s}{2}} \sup_u \mathbb{P}(\mathcal{M}_{u,\omega}(p, \mathcal{X}, T_k) > T_k^\alpha) < p_0/12. \tag{2.3.38}$$

In an analogous way we can estimate (2.3.17) and (2.3.18). It follows that for all $E \in I$ we have

$$\limsup_{k\to\infty}\sup_y \mathbb{P}\left(\frac{1}{L_k^\theta} < \|\Gamma_{y,L_k} R_{\omega,y,L_k}(E)\chi_{y,L_k/3}\|_{L_k}\right) < p_0. \tag{2.3.39}$$

Since $0 < p_0 < 1$ is arbitrary, we conclude that (2.3.3) holds for each $E \in I$.

□

Proof of Lemma 2.3.3 . Let $\omega \in \{\omega : \mathcal{M}_{u,\omega}(p, \mathcal{X}, T) \le T^\alpha\}$. For a given compact subinterval $I \subset J$, $M > 0$ and $L = L(\epsilon, I)$ as in (2.3.12), we set

$$A_{u,\omega,M,I} = \{E \in I : \ \|\langle X - u\rangle^{p/2} R_\omega(E + i\epsilon)\mathcal{X}(H_\omega)\chi_u\|_2^2 \le M\epsilon^{-(\alpha+1)}\}.$$

We have, using [GK04, Lemma 6.3]

$$\begin{aligned}
|I \setminus A_{u,\omega,M,I}| &\le \frac{1}{M\epsilon^{-(\alpha+1)}}\int_{\mathbb{R}} \|\langle X - u\rangle^{p/2} R_\omega(E + i\epsilon)\mathcal{X}(H_\omega)\chi_u\|_2^2 dE \\
&= \frac{2\pi}{MT^{\alpha+1}}\int_0^\infty e^{-2t/T}\|\langle X - u\rangle^{p/2} e^{-itH_\omega}\mathcal{X}(H_\omega)\chi_u\|_2^2 dt \\
&= \frac{\pi}{MT^\alpha}\mathcal{M}_{u,\omega}(p, \mathcal{X}, T) \\
&\le \frac{\pi}{M},
\end{aligned} \tag{2.3.40}$$

where the last bound is uniform on u and ω.

Thus, for an $E \in I$ fixed either $E \in A_{u,\omega,M,I}$ in which case we have

$$\begin{aligned}
\|\Gamma_{y,L} R_\omega(E + i\epsilon)\mathcal{X}(H_\omega)\chi_u\| &\le C_{p,d} L^{-p/2}\|\langle X - u\rangle^{p/2} R_\omega(E + i\epsilon)\mathcal{X}(H_\omega)\chi_u\|_2 \\
&\le C_{p,d} L^{-p/2} M^{1/2}\epsilon^{-(\alpha+1)/2}
\end{aligned} \tag{2.3.41}$$

or else, $E \in I \setminus A_{u,\omega,M,I}$, so by (2.3.40) there exists $E_{u,\omega} \in A_{u,\omega,M,I}$ such that

$$|E - E_{u,\omega}| \le \frac{\pi}{M}$$

and therefore, by the resolvent identity and the definition of $A_{u,\omega,M,I}$,

$$\begin{aligned}
\|\Gamma_{y,L} R_\omega(E + i\epsilon)\mathcal{X}(H_\omega)\chi_u\| &\le \|\Gamma_{y,L} R_\omega(E_{u,\omega} + i\epsilon)\mathcal{X}(H_\omega)\chi_u\| \\
&\quad + |E - E_{u,\omega}|\|R_\omega(E + i\epsilon)\|\|R_\omega(E_{u,\omega} + i\epsilon)\| \\
&\le C_{p,d} L^{-p/2} M^{1/2}\epsilon^{-(\alpha+1)/2} + \frac{\pi}{M\epsilon^2}.
\end{aligned} \tag{2.3.42}$$

Now, for p such that $p > p(\theta, \gamma, d, \alpha, s)$ we have

$$2(\theta + \gamma + d) < p - 6\theta - \gamma - (1 + \alpha)(\theta s + d)/s, \tag{2.3.43}$$

so if we set

$$M = L^{6\theta+\gamma}, \tag{2.3.44}$$

and recall

$$\epsilon^{-(1+\alpha)/2} = C_{p_0,Q_I} L^{(1+\alpha)(\theta s+d)/2s}, \tag{2.3.45}$$

we obtain, for L big enough depending on $d, I, p, \alpha, \theta, \gamma, s, p_0, Q_I$,

$$\begin{aligned} C_{p,d} L^{-p/2} M^{1/2} \epsilon^{-(\alpha+1)/2} &= C_{p,d,Q_I,p_0} L^{-(p/2-(6\theta+\gamma)/2-(1+\alpha)(\theta s+d)/2s)} \\ &< \frac{1}{4L^{(\theta+\gamma+d)}} \end{aligned} \tag{2.3.46}$$

and

$$\frac{\pi}{M\epsilon^2} = C'_{p_0,Q_I} L^{6\theta+2\gamma-2(\theta s+d)/s} < \frac{1}{4L^{(\theta+\gamma+d)}}. \tag{2.3.47}$$

Inserting this in (2.3.42) proves the lemma.

□

2.4 The case of a weak Wegner estimate: Proof of Theorem 2.1.11

Proof of Theorem 2.1.11. For the polynomial weak Wegner estimate (2.1.19), in order to start the MSA we need a different ILSE than (2.1.5) (see eq. 5.7, Theorem 5.6 in [GK01]). It is enough to show that, for each $E \in J$ there is some $\theta > \eta$ such that for L_0 big enough

$$\limsup_{L\to\infty} \inf_{y\in\mathbb{Z}^d} \mathbb{P}\left(\|\Gamma_{y,L} R_{\omega,y,L}(E) \chi_{y,L/3}\|_{y,L} \le e^{-L^\theta} \right) \ge 1 - \frac{1}{L^\xi}, \tag{2.4.1}$$

Let $E \in J$, $\theta > d$ and $L \in 6\mathbb{N}$. We proceed as in [GK04] and we start by estimating

$$\sup_{y\in\mathbb{Z}^d} P_{E,y,L} := \sup_{y\in\mathbb{Z}^d} \mathbb{P}\left(\|\Gamma_{y,L} R_{\omega,y,L}(E) \chi_{y,L/3}\|_{y,L} > e^{-L^\theta} \right), \tag{2.4.2}$$

By adapting the proof of [GK04, Lemma 6.3] (see Lemma 2.3.1) to the weak Wegner estimate we obtain the following lemma

Lemma 2.4.1. *Let H_ω be a Schrödinger operator as stated above, satisfying a weak Wegner estimate on an open interval J. For $E \in J$ and $p_0 > 0$. there exists $\mathcal{L} = \mathcal{L}(p_0, d, E, q, \theta, \eta)$, such that for $L > \mathcal{L}$, and taking*

$$\epsilon = e^{-3L^\theta}, \tag{2.4.3}$$

we have

$$\mathbb{P}\left(\|\Gamma_{y,L} R_{\omega,L}(E) \chi_{y,L/3}\|_L > e^{-L^\theta} \right) \le \mathbb{P}\left(\|\Gamma_{y,L} R_\omega(E+i\epsilon) \chi_{y,L/3}\| > \frac{e^{-3L^\beta}}{2} \right) + 2p_0. \tag{2.4.4}$$

To estimate the $\sup_y$ of the r.h.s. of (2.4.4) we decompose in energy using a cutoff function $\mathcal{X} \in C^{\infty}_{c,+}(\mathbb{R})$ such that $\mathcal{X} \equiv 1$ on J, we are left with the sum

$$\sup_y \mathbb{P}\left(\frac{e^{-3L^\beta}}{4} < \|\Gamma_{y,L} R_\omega(E+i\epsilon)\mathcal{X}(H_\omega)\chi_{y,L/3}\|\right) \tag{2.4.5}$$

$$+\sup_y \mathbb{P}\left(\frac{e^{-3L^\beta}}{4} < \|\Gamma_{y,L} R_\omega(E+i\epsilon)(1-\mathcal{X}(H_\omega))\chi_{y,L/3}\|\right). \tag{2.4.6}$$

Using Chebyshev's inequality we can estimate (2.4.6) by

$$4e^{3L^\beta}\mathbb{E}\left(\|\Gamma_{y,L} R_\omega(E+i\epsilon)(1-\mathcal{X}(H_\omega))\chi_{y,L/3}\|\right) = 4e^{3L^\theta} E\left(\|\Gamma_{y,L} f_{E,\epsilon}(H_\omega)\chi_{y,L/3}\|\right), \tag{2.4.7}$$

where

$$f_{E,\epsilon}(u) = (u-(E+i\epsilon)^{-1}))(1-\mathcal{X}(u)) \tag{2.4.8}$$

is a bounded, infinitely differentiable function on the real line (see [GK04]). Moreover, choosing $\mathcal{X}$ conveniently, $f_{E,\epsilon}$ is of weighted Gevrey-class $G_1^a(\mathbb{R})$, with a arbitrarily close to 1, and by results on the subexponential decay of Gevrey-type functions of Schrödinger operators in [BGK04, Theorem 1.4-(i)], for each a' with $a' > a$ there exist constants C_1 and c_2 depending only on the function $f_{E,\epsilon}$ such that $\mathbb{P}$-a.e

$$\|\chi_y f_{E,\epsilon}(H_\omega)\chi_x\| \leq C_1 e^{-c_2|x-y|^{1/a'}}, \quad \text{for all } x, y \in \mathbb{R}^d. \tag{2.4.9}$$

Choosing a' conveniently, plus the fact that $\mathrm{dist}(\Upsilon_{y,L}, \Lambda_{y,L/3}) = \frac{L}{3} - \frac{3}{2}$ we get that for L big enough, depending on θ, C_1, c_2, p_0,

$$\sup_y 4e^{3L^\theta} E\left(\|\Gamma_{y,L} f_{E,\epsilon}(H_\omega)\chi_{y,L/3}\|\right) < p_0 \tag{2.4.10}$$

This yields,

$$P_{E,y,L} \leq 4e^{3L^\theta}\mathbb{E}\left(\|\Gamma_{y,L} R_\omega(E+i\epsilon)\mathcal{X}(H_\omega)\chi_{y,L/3}\|\right) + 3p_0 \tag{2.4.11}$$

$$\leq 4e^{3L^\theta} \sum_{u\in\Lambda_{y,L/3}} \mathbb{E}\left(\|\Gamma_{y,L} R_\omega(E+i\epsilon)\mathcal{X}(H_\omega)\chi_u\|\right) + 3p_0 \tag{2.4.12}$$

$$\tag{2.4.13}$$

Now, [GK04, Prop. 6.1] applied to the subexponential moment gives

$$\mathcal{M}_u(\sigma,\zeta,\mathcal{X},T) = \frac{1}{\pi T}\int_{\mathbb{R}} \mathbb{E}\left(\|e^{\frac{\sigma}{2}|X-u|^\zeta} R_\omega(E+i\frac{1}{T})\mathcal{X}(H_\omega)\chi_u\|_2^2\right) dE \tag{2.4.14}$$

So the condition on the growth of the subexponential moment :

$$\liminf_{T\to\infty} \frac{1}{T^\alpha}\mathbb{E}\left(\mathcal{M}_{u,\omega}(\sigma,\zeta,\mathcal{X},T)\right) < \infty, \tag{2.4.15}$$

implies, by 2.4.14, that there exists a sequence $\epsilon_k \to 0^+$ and a constant $C > 0$ such that

$$\epsilon_k^{\alpha+1} \int_{\mathbb{R}} \mathbb{E}\left(\|e^{\frac{\sigma}{2}|X-u|^\zeta} R_\omega(E+i\epsilon_k)\mathcal{X}(H_\omega)\chi_u\|_2^2\right) dE < C. \tag{2.4.16}$$

We define the set $A_{k,u,M,I}$ by

$$A_{k,u,M,I} = \left\{E \in I :\ \mathbb{E}\left(\|e^{\frac{\sigma}{2}|X-u|^\zeta} R_\omega(E+i\epsilon_k)\mathcal{X}(H_\omega)\chi_u\|_2^2\right) \le M\epsilon_k^{-(\alpha+1)}\right\}. \tag{2.4.17}$$

By (2.4.16) we have that , uniformly in u ,

$$|I \setminus A_{k,u,M,I}| < \frac{C}{M}. \tag{2.4.18}$$

Thus, for $E \in I$ fixed, either $E \in A_{k,u,M,I}$ in which case we have (for simplicity, in the following we write $L = L_k, \epsilon = \epsilon_k$)

$$\begin{aligned}
&4e^{3L^\theta}\mathbb{E}\left(\|\Gamma_{y,L}R_\omega(E+i\epsilon)\mathcal{X}(H_\omega)\chi_u\|\right) \le \\
&\le 4e^{3L^\theta}\mathbb{E}\left(\|\Gamma_{y,L}e^{-\frac{\sigma}{2}|X-u|^\zeta}e^{\frac{\sigma}{2}|X-u|^\zeta}R_\omega(E+i\epsilon)\mathcal{X}(H_\omega)\chi_u\|\right) \\
&\le 4e^{3L^\theta}e^{-\frac{\sigma}{2}\left(\frac{L}{4}\right)^\zeta}\mathbb{E}\left(\|e^{\frac{\sigma}{2}|X-u|^\zeta}R_\omega(E+i\epsilon)\mathcal{X}(H_\omega)\chi_u\|_2\right) \\
&\le 4e^{3L^\theta-\frac{\sigma}{2}\left(\frac{L}{4}\right)^\zeta}\mathbb{E}\left(\|e^{\frac{\sigma}{2}|X-u|^\zeta}R_\omega(E+i\epsilon)\mathcal{X}(H_\omega)\chi_u\|_2^2\right)^{1/2} \\
&\le 4e^{3L^\theta-\frac{\sigma}{2}\left(\frac{L}{4}\right)^\zeta}M^{1/2}\epsilon^{-(\alpha+1)}. && (2.4.19)
\end{aligned}$$
$$(2.4.20)$$

or else, $E \in I \setminus A_{k,u,M,I}$, so by (2.4.18) there exists $E_u \in A_{u,I,M,\epsilon_k}$ such that

$$|E - E_u| \le \frac{\pi}{M}$$

and so, by the resolvent identity and the definition of $A_{k,u,M,I}$,

$$\begin{aligned}
&4e^{3L^\theta}\mathbb{E}\left(\|\Gamma_{y,L}R_\omega(E+i\epsilon)\mathcal{X}(H_\omega)\chi_u\|\right) && (2.4.21)\\
&\le 4e^{3L^\theta}\mathbb{E}\left(\|\Gamma_{y,L}R_\omega(E_u+i\epsilon)\mathcal{X}(H_\omega)\chi_u\|\right) + \\
&\qquad |E-E_u|\mathbb{E}\left(\|R_\omega(E+i\epsilon\|\|R_\omega(E_u+i\epsilon)\|\right) \\
&\le 4e^{3L^\theta-\frac{\sigma}{2}\left(\frac{L}{4}\right)^\zeta}M^{1/2}\epsilon^{-(\alpha+1)} + \frac{\pi}{M\epsilon^2}. && (2.4.22)
\end{aligned}$$
$$(2.4.23)$$

Therefore,

$$P_{E,y,L} \le \frac{4}{3^d}L^d e^{3L^\theta-\frac{\sigma}{2}\left(\frac{L}{4}\right)^\zeta}M^{1/2}\epsilon^{-(1+\alpha)/2} + \frac{2\pi}{3^d}\frac{L^d e^{3L^\theta}}{M\epsilon^2} + 3p_0. \tag{2.4.24}$$
$$(2.4.25)$$

By choosing

$$M = e^{(9+\alpha+1)L^{\theta}}, \tag{2.4.26}$$

and recalling (2.4.3), this immediately yields that the second term in the r.h.s. is smaller than p_0 for L big enough depending on θ, p_0, d, α. As for the remaining term,

$$\frac{4}{3^d}\left(\frac{2\pi}{3^d}\right)^{1/2} L^{2d} e^{-\frac{\sigma}{2}\left(\frac{L}{4}\right)^{\zeta}} e^{\left(3+\frac{9}{2}+2(\alpha+1)\right)L^{\theta}} \tag{2.4.27}$$

if $\zeta > \theta$, any $\sigma > 0$ will give a subexponential decay for L big enough depending on $\theta, d, \sigma, \zeta, p_0$. By assuming $\zeta > \eta$ we can always find such θ, with $\theta > \eta$.

We have proved there exists $\mathcal{L} = \mathcal{L}(\theta, p_0, d, \zeta, \eta, \alpha, \sigma, E, q)$ such that for $L > \mathcal{L}$ we have

$$\sup_y P_{E,y,L} < 5p_0. \tag{2.4.28}$$

Since p_0 is arbitrary, we get the desired result. □

Remark 2.4.2. In the case of the exponentially weak Wegner estimate, valid in $d = 1$, in order to start the Bootstrap MSA the probability in the r.h.s. of 2.4.1 needs to be $\geq 1 - e^{-L_0^{\xi}}$ (see [GK01, eq. 5.9] and Remark 3.13 therein). The proof of Theorem 2.1.11 for this case follows in the same way as for the polynomial Wegner estimate, with some minor modifications, yielding subexponential decay of probabilities of bad events. Therefore, in this case, we have the equivalence $\Sigma_{SEDL} = \Sigma_{MSA}$.

Chapter 3

Delone–Anderson Operators I: The magnetic case

Contents

3.1 The model and main result

In this chapter, we study a particular non-ergodic model, that represents a particle moving in dimension 2 under the influence of a constant magnetic field in a random medium, where impurities, or obstacles, are centered in points of a Delone set. If the Delone set is periodic, we retrieve the standard (ergodic) Anderson operator. In this section we introduce the model and in the following section we prove it satisfies a uniform Wegner estimate. In the last section we prove it also satisfies an initial length scale estimate. Then, by applying results from Chapter 2, we show the existence of a metal-insulator transition in each Landau band, as expected from the ergodic case. The existence of a mobility edge is the result of complementary spectral regions of dynamical localization and delocalization. We end our study by proving that these results are not trivial, that is, we prove the almost sure existence of spectrum in the regions where we prove dynamical localization. More precisely, we prove that if the spectrum of the operator has spectral gaps, they cannot be larger than $B^{-1/2}$, where B is the intensity of the magnetic field. Sections 3.2 and 3.3 are contained in the article "Characterization of the Anderson metal-insulator transition for non ergodic operators and application" published in Annales Henri Poincaré [RM12]. These results were announced in [GRM11].

Definition 3.1.1. A subset D of $\mathbb{R}^d$ is called an *(r,R)-Delone set* if it is

- uniformly discrete: there exist a real $r > 0$ such that for any cube Λ_r, $\sharp(D \cap \Lambda_r) \leq 1$, and
- relatively dense: there exists a real $R \geq r > 0$ such that for any cube Λ_R, $\sharp(D \cap \Lambda_R) \geq 1$,

where $\sharp$ stands for cardinality.

Remark 3.1.2. *Note that in a (r, R)-Delone set there exists a minimal distance between any two points, $r/2$, and a maximal distance between neighbors, $\sqrt{d}R$. Lattices and the set of vertices of a Penrose tiling are particular cases of Delone sets (see Fig. 1.1.*

We consider a random magnetic Schrödinger operator of the form $H_\omega = H_B + \lambda V_\omega$ on $\mathrm{L}^2(\mathbb{R}^2)$. The background Landau Hamiltonian H_B is defined by

$$H_B = (-i\nabla - \mathbf{A})^2 \quad \text{with } \mathbf{A} = \frac{B}{2}(x_2, -x_1), \tag{3.1.1}$$

where $\mathbf{A}$ is the vector potential, the constant $B > 0$ is the strength of the magnetic field, and the random potential represents impurities placed in a Delone set, that is,

$$V_\omega(x) = \sum_{\gamma \in D} \omega_\gamma u(x - \gamma), \tag{3.1.2}$$

We make the following assumptions on the Delone–Anderson potential:

(v1) The *single-site potential* u is a measurable function such that $\| \sum_{\gamma \in D} u(\cdot - \gamma)\|_\infty = 1$, it has compact support and satisfies

$$u^- \chi_{0,\epsilon_u} \leq u \leq u^+ \chi_{0,\delta_u}, \tag{3.1.3}$$

for some constants $0 < \epsilon_u \leq \delta_u < r < \infty$ and $0 < u^- \leq u^+ < \infty$.

(v2) $(\omega_\gamma)_{\gamma \in D}$ is a family of i.i.d. random variables, with probability distribution μ of bounded and continuous density ρ such that

$$\rho_+ := \|\rho\|_\infty < \infty, \tag{3.1.4}$$

$$0 \in \operatorname{supp} \rho \subset [-m_0, M_0], \tag{3.1.5}$$

where $0 \leq m_0 < \infty$, $0 < M_0 < \infty$. We define the global modulus of continuity of μ as

$$s(\epsilon) = \sup_{E \in \mathbb{R}} \mu([E - \epsilon/2, E + \epsilon/2]) \tag{3.1.6}$$

Since μ is an absolutely continuous probability distribution, $s(\epsilon) \leq \rho_+ \epsilon$.

(uc) $\delta_u < r/10$, i.e. u has compact support contained in $B(0, r/10)$. This implies that for $i, j \in D$ with $i \neq j$, $\operatorname{supp} u_i \cap \operatorname{supp} u_j = \emptyset$, where we use the notation $u_i = u(\cdot - i)$ for $i \in \mathbb{R}^2$.

(u0) $\|u\|_\infty = 1$ and $u(0) = 1$.

Under these assumptions V_ω is a bounded scalar potential jointly measurable in both $\omega \in \Omega$ and $x \in \mathbb{R}^d$, and so the mapping $\omega \mapsto H_\omega$ is measurable.

Remark 3.1.3. We can also treat the case where the random variables ω_γ are independent but not identically distributed. Here, the global modulus of continuity s is defined as $s(\epsilon) = \sup_{\gamma\in D}\sup_{E\in\mathbb{R}} \mu_\gamma([E-\epsilon/2, E+\epsilon/2])$. In such case a disorder assumption is needed for the distributions μ_γ, namely, $s(\epsilon) < \infty$, $\rho_+ = \sup_\gamma \|\rho_\gamma\|_\infty < \infty$ and that for $\epsilon > 0$, there exist constants $C, \tau > 0$ such that $\mu_\gamma([E-\epsilon, E+\epsilon]) \geq C\epsilon^\tau$ for all $\gamma \in D$.

The spectrum of H_B is pure point and consists of a sequence of infinitely degenerate eigenvalues, the Landau levels $\{B_n = (2n+1)|B|;\ n = 0, 1, ...\}$, with associated orthogonal projection operators Π_n. As the spectrum is independent of the sign of B, we will always assume $B > 0$.

We denote the spectrum of the operator $H_{B,\lambda,\omega}$ by $\sigma_{B,\lambda,\omega}$. By perturbation theory [K, Theorem V.4.10] we know that for each $\omega \in \Omega$,

$$\sigma_{B,\lambda,\omega} \subset \bigcup_{n=0}^{\infty} \mathcal{B}_n(B, \lambda),$$

where $\mathcal{B}_n(B,\lambda) = [B_n - \lambda m_0, B_n + \lambda M_0]$ is called the *n-th Landau band*. Moreover, by a Borel–Cantelli argument, for almost every $\omega \in \Omega$,

$$\sigma_B \subset \sigma_{B,\lambda,\omega}, \tag{3.1.7}$$

where σ_B is the spectrum of the free Landau operator. We also show that there exists almost surely spectrum near the band edges so our results are not empty (see Section 3.3.3).

For B fixed λ is small enough such that

$$\lambda(m_0 + M_0) < 2B, \tag{3.1.8}$$

i.e., the Landau bands $\mathcal{B}_n(B,\lambda)$ are disjoint and hence the open intervals

$$\mathcal{G}_n(B,\lambda) =]B_n + \lambda M_0, B_{n+1} - \lambda m_0[, \qquad n = 0, 1, 2, ..., \tag{3.1.9}$$

are nonempty spectral gaps for $H_{B,\lambda,\omega}$.

We define the magnetic translations U_a for $a \in \mathbb{R}^2$ and $\varphi \in \mathcal{C}_0^\infty(\mathbb{R}^2)$, by

$$U_a\varphi(x) = e^{-i\frac{B}{2}(x_2a_1 - x_1a_2)}\varphi(x-a), \tag{3.1.10}$$

obtaining a projective unitary representation of $\mathbb{R}^2$ on $\mathrm{L}^2(\mathbb{R}^2)$:

$$U_aU_b = e^{i\frac{B}{2}(a_2b_1 - a_1b_2)}U_{a+b} = e^{iB(a_2b_1-a_1b_2)}U_bU_a, \quad a, b \in \mathbb{R}^2. \tag{3.1.11}$$

We then have $U_aH_BU_a^* = H_B$ for all $a \in \mathbb{R}^2$.

We now define finite volume operators following [GKS07]. For $B > 0$, we set

$$K_B = \min\left\{k \in \mathbb{N} : k \geq \sqrt{\frac{B}{4\pi}}\right\} \quad \text{and} \quad L_B = K_B\sqrt{\frac{B}{4\pi}}. \tag{3.1.12}$$

We denote $\mathbb{N}_B = L_B\mathbb{N}$, $\tilde{\mathbb{N}}_B = \mathbb{N}_B \cup \{\infty\}$ and $\mathbb{Z}_B^2 = L_B\mathbb{Z}^2$.

We consider squares $\Lambda_L(x)$ with $L \in \mathbb{N}_B$ and $x \in \mathbb{R}^2$, and identify them with the torii $\mathbb{T}_{L,x} := \mathbb{R}^2/(L\mathbb{Z}^2 + x)$. We denote by $\chi_{x,L}$ the characteristic function of the cube $\Lambda_L(x)$ and for $\tilde{x} \in \Lambda_L(x)$ and $\tilde{L} < L$ we denote by $\hat{\Lambda}_{\tilde{L}}(\tilde{x})$ and $\hat{\chi}_{\tilde{x},\tilde{L}}$ the cube and characteristic function in $\mathbb{T}_{L,x}$.

For the first order differential operator $\mathbf{D}_B = (-i\nabla - \mathbf{A})$ restricted to $\mathcal{C}_c^\infty(\Lambda_L(x))$ we take its closed, densely defined extension $\mathbf{D}_{B,x,L}$ from $\mathrm{L}^2(\Lambda_L(x))$ to $\mathrm{L}^2(\Lambda_L(x);\mathbb{C}^2)$, with periodic boundary conditions and then set $H_{B,x,L} = \mathbf{D}^*_{B,x,L}\mathbf{D}_{B,x,L}$.

We are left with the operator $H_{B,\lambda,\omega,x,L}$ acting on $\mathrm{L}^2(\Lambda_L(x))$ defined by

$$H_{B,\lambda,\omega,x,L} = H_{B,x,L} + \lambda V_{\omega,x,L}. \tag{3.1.13}$$

where $V_{\omega,x,L}$ is defined by

$$V_{\omega,x,L}(\cdot) = \sum_{\gamma \in D \cap \Lambda_{L-\delta_u}(x)} \omega_\gamma u(\cdot - \gamma). \tag{3.1.14}$$

and we denote by $\tilde{V}_{x,L}$ the potential defined by

$$\tilde{V}_{x,L}(\cdot) = \sum_{\gamma \in \tilde{\Lambda}_{L-\delta_u}(x)} u(\cdot - \gamma), \tag{3.1.15}$$

where $\tilde{\Lambda}_L(x) = D \cap \Lambda_L(x)$.

Since $H_{B,x,L}$ has a compact resolvent, its spectrum consists in the Landau Levels but now with finite multiplicity. We denote by $\Pi_{n,x,L}$ the orthogonal projection associated to the n-th Landau level, $\Pi^\perp_{n,x,L}$ its orthogonal complement, define $P_{B,\lambda,\omega,x,L}(J) = \chi_J(H_{B,\lambda,\omega,x,L})$ for $J \subset \mathbb{R}$ a Borel set and write $R_L(z) = (H_{B,\lambda,\omega,x,L} - z)^{-1}$ for the resolvent operator of $H_{B,\lambda,\omega,x,L}$. We drop the subscript x from the notation when the results are uniform with respect to the center of the box Λ_L.

This operator satisfies the compatibility conditions [GKS07, Eq. 4.2]: If $\varphi \in \mathcal{D}(\mathbf{D}_{B,x,L})$ with $\operatorname{supp}\varphi \subset \Lambda_{L-\delta_u}(x)$, then $\mathcal{I}_{x,L}\varphi \in \mathcal{D}(\mathbf{D}_B)$ and

$$\begin{gathered}\mathcal{I}_{x,L}\mathbf{D}_{B,x,L}\varphi = \mathbf{D}_B\mathcal{I}_{x,L}\varphi,\\ \mathcal{I}_{x,L}\chi_{x,L-\delta_u}V_{\omega,x,L} = \chi_{x,L-\delta_u}V_\omega,\end{gathered} \tag{3.1.16}$$

where $\mathcal{I}_{x,L} : \mathrm{L}^2(\Lambda_L(x)) \to \mathrm{L}^2(\mathbb{R}^2)$ is the canonical injection

$$\mathcal{I}_{x,L}\varphi(y) = \begin{cases} \varphi(y) & \text{if } y \in \Lambda_L(x) \\ 0 & \text{otherwise.} \end{cases}$$

From this we have

$$\mathcal{I}_{x,L}H_{B,\lambda,\omega,x,L}\varphi = H_{B,\lambda,\omega}\mathcal{I}_{x,L}\varphi,$$

that is, the finite volume operators $H_{B,\lambda,\omega,x,L}$ agree with $H_{B,\lambda,\omega}$ inside the square $\Lambda_L(x)$.

However, $H_{B,\lambda,\omega,x,L}$ does not satisfy the covariance condition (1.1.3) so we have *a priori*

$$H_{B,\lambda,\omega,x,L} \neq U_x H_{B,\lambda,\tau_{-x}(\omega),0,L}U_x^*,$$

where U_x is the magnetic translation (3.1.10) seen as a unitary map from $\mathrm{L}^2(\Lambda_L(0))$ to $\mathrm{L}^2(\Lambda_L(x))$ and τ_x is the translation defined as $\tau_x(\omega_\gamma) = \omega_{\gamma-x}$ for $x \in \mathbb{R}^2$.

We aim to prove for this model the existence of complementary regions of dynamical localization and delocalization in the spectrum and therefore, the existence of a dynamical transition energy. By doing this we extend known results for ergodic random Landau Hamiltonians [CH96, GK03, GKS07, GKS09] to non-ergodic ones.

Theorem 3.1.4. *Let $H_{B,\lambda,\omega}$ be the Delone-Landau Hamiltonian, satisfying in particular condition* (3.1.8)*. Then for any $n = 0, 1, 2, ...$ there exists a positive constant $B(n)$, depending on parameters of the model, such that for any $B > B(n)$, $H_{B,\lambda,\omega}$ exhibits almost surely an Anderson metal-insulator transport transition in the n-th Landau band.*

3.2 A uniform Wegner estimate for a Delone–Anderson perturbation of the Landau Hamiltonian

Theorem 3.2.1. *For $d = 2$, let H_0 be the Landau Hamiltonian with constant magnetic field $B > 0$ fixed. For any bounded interval $I \in \mathbb{R}$ there exist constants $Q_W = Q_W(B, \lambda, R, r, I, u, m_0, M_0)$, $\eta_{B,\lambda,J} \in]0, 1]$ and a finite scale $\mathcal{L}_*(B, \lambda, I, R)$ such that for every compact subinterval $J \subset I$, with $|J| < \eta_{B,\lambda,J}$ and $L > \mathcal{L}_*$, we have*

$$\sup_{x\in\mathbb{R}^d} \mathbb{E}\{ \ tr\ P_{\lambda,\omega,x,L}(J)\} \leq Q_W s(|J|)L^d. \tag{3.2.1}$$

Remark 3.2.2. *We point out that Theorem 3.2.1 provides a Wegner estimate at all energies, in particular, at Landau levels. Compare this result to Theorem 3.3.2. Both these results are necessary to prove the transport transition of Theorem 3.1.4. Theorem 3.3.2 is used to prove localization at the band edges, while Theorem 3.2.1 is needed to apply Theorem 2.1.6.*

Since the results are uniform in x, we state them for $x = 0$, λ fixed and for simplicity we omit these subscripts from the notation.

For the proof we follow [CHK07], based on [CHK03], plus [GKS07]. For an arbitrary $E_0 \in \mathbb{R}$, with J and $\tilde{J}$ closed bounded intervals centered on E_0 such that $J \subset \tilde{J}$, $|J| < 1$, $d_J > 0$, we decompose $\mathbb{E}\{\mathrm{tr}\, P_{\omega,L}(J)\}$ with respect to the free spectral projector of $\tilde{J}$,

$$\mathrm{tr}\, P_{\omega,L}(J) = \mathrm{tr}\, P_{\omega,L}(J)\Pi_{n,L} + \mathrm{tr}\, P_{\omega,L}(J)\Pi_{n,L}^{\perp}. \tag{3.2.2}$$

The key step in estimating the first term of the r.h.s is to prove a positivity estimate as in [CHK07, Theorem 2.1]. In order to obtain this estimate in the case of the Landau Hamiltonian, we need some preliminary lemmas.

Lemma 3.2.3. *Using the notations above, there exists a positive finite constant $C_n(B, u, R)$, so that*

$$\Pi_{n,L}\tilde{V}_{x,L}\Pi_{n,L} \geq C_n(B, u, R)\Pi_{n,L}. \tag{3.2.3}$$

Proof. From [CHKR04] we have that for $n \in \mathbb{N}$, $\tilde{R} > 0$, for each $0 < \epsilon < \tilde{R}$, $\kappa > 1$ and $\eta > 0$ there exists a constant $C_0 = C_{0,n,\epsilon,\tilde{R},\eta} > 0$ such that

$$\Pi_n \chi_{0,\epsilon} \Pi_n \geq C_0 (\Pi_n \chi_{0,\tilde{R}} \Pi_n - \eta \Pi_n \chi_{0,\kappa\tilde{R}} \Pi_n). \tag{3.2.4}$$

Because of the invariance of H_B under the magnetic translations (3.1.10) we have that the projections Π_n commute with these unitary operators, which in turn gives, for an arbitrary $x \in \mathbb{R}^2$,

$$U_x \Pi_n \chi_{0,\epsilon} \Pi_n U_x^* \geq C_0 U_x (\Pi_n \chi_{0,\tilde{R}} \Pi_n - \eta \Pi_n \chi_{0,\kappa\tilde{R}} \Pi_n) U_x^* \tag{3.2.5}$$

$$\Pi_n U_x \chi_{0,\epsilon} U_x^* \Pi_n \geq C_0 (\Pi_n U_x \chi_{0,\tilde{R}} U_x^* \Pi_n - \eta \Pi_n U_x \chi_{0,\kappa\tilde{R}} U_x^* \Pi_n) \tag{3.2.6}$$

$$\Pi_n \chi_{x,\epsilon} \Pi_n \geq C_0 (\Pi_n \chi_{x,\tilde{R}} \Pi_n - \eta \Pi_n \chi_{x,\kappa\tilde{R}} \Pi_n), \tag{3.2.7}$$

since conjugation by unitary operators is a positivity preserving operation.

Now, we recall [GKS07, Lemma 5.3] (which is independent of V and, therefore, D).

Lemma 3.2.4. *Fix $B > 0$, $n \in \mathbb{N}$, $\tilde{R} > 0$, $0 < \epsilon < \tilde{R}$ and $\eta > 0$. If $\kappa > 1$ and $L \in \mathbb{N}_B$ (defined as in* (3.1.12)*) are such that $L > 2(L_B + \kappa\tilde{R})$ then for all $\tilde{x} \in \Lambda_L(x)$, we have*

$$\Pi_{n,L} \hat{\chi}_{\tilde{x},\epsilon} \Pi_{n,L} \geq C_0 \Pi_{n,L} (\hat{\chi}_{\tilde{x},\tilde{R}} - \eta \hat{\chi}_{\tilde{x},\kappa\tilde{R}}) \Pi_{n,L} + \Pi_{n,L} \mathcal{E}_{n,\tilde{x},L} \Pi_{n,L}, \tag{3.2.8}$$

where $C_0 = C_{0;n,B,\epsilon,\tilde{R},\eta} > 0$ is a constant as before and the error operator $\mathcal{E}_{n,\tilde{x},L}$ satisfies

$$\|\mathcal{E}_{n,\tilde{x},L}\| \leq C_{n,B,\epsilon,R,\eta} e^{-m_{n,B} L}, \tag{3.2.9}$$

for some positive constant $m_{n,B}$.

Now, by (4.1.2) we have

$$\tilde{V}_{x,L}(\cdot) = \sum_{\gamma \in \tilde{\Lambda}_{L-\delta_u}(x)} u(\cdot - \gamma) \geq u^- \sum_{\gamma \in \tilde{\Lambda}_{L-\delta_u}(x)} \hat{\chi}_{\gamma,\epsilon_u}. \tag{3.2.10}$$

We fix $\tilde{R} > 2R + \delta_u$, in which case

$$\sum_{\gamma \in \tilde{\Lambda}_{L-\delta_u}(x)} \hat{\chi}_{\gamma,\tilde{R}} \geq \chi_{x,L}. \tag{3.2.11}$$

Now fix $\kappa > 1$ and pick $\eta > 0$ such that

$$\eta \sum_{\gamma \in \tilde{\Lambda}_{L-\delta_u}(x)} \hat{\chi}_{\gamma,\kappa\tilde{R}} \leq \frac{1}{2} \chi_{x,L}. \tag{3.2.12}$$

It follows from Lemma 3.2.4, (3.2.11) and (3.2.12) that

$$\Pi_{n,L}\tilde{V}_{x,L}\Pi_{n,L} \geq u^- C_0 \sum_{\gamma\in\tilde{\Lambda}_{L-\delta_u}(x)} \Pi_{n,L}(\hat{\chi}_{\gamma,\tilde{R}} - \eta\hat{\chi}_{\gamma,\kappa\tilde{R}})\Pi_{n,L} + \Pi_{n,L}\mathcal{E}_{n,L}\Pi_{n,L} \tag{3.2.13}$$

$$\geq \frac{u^- C_0}{2}\Pi_{n,L} + \Pi_{n,L}\mathcal{E}_{n,L}\Pi_{n,L} \tag{3.2.14}$$

$$\geq C_1\Pi_{n,L}, \tag{3.2.15}$$

for $L \geq L^*$ for some $L^* = L^*_{n,B,\epsilon,R,\kappa,\eta} < \infty$ and $C_1 = \frac{u^- C_0}{4}$, since the error operator

$$\Pi_{n,L}\mathcal{E}_{n,L}\Pi_{n,L} = \Pi_{n,L} \sum_{\gamma\in\tilde{\Lambda}_{L-\delta_u}(x)} \mathcal{E}_{n,\gamma,L}\Pi_{n,L}$$

by (3.2.9), satisfies

$$\|\mathcal{E}_{n,L}\| \leq L^2 C_{n,B,\epsilon,R,\eta} e^{-m_{n,B}L}.$$

□

Finally we recall [CHK07, Lemma 2.1],

Lemma 3.2.5. *Suppose that T is a trace class operator independent of ω and u, the single site potential* (4.1.2)*. We then have*

$$\mathbb{E}\{\mathrm{tr}\, P_{\omega,L}(J)u_i T u_j\} \leq 8s(|J|)\|u_i T u_j\|_1. \tag{3.2.16}$$

where we use the notation $u_i = u(x-i)$, $i \in \mathbb{R}^2$.

Proof of Theorem 3.2.1. Using the preliminary lemmas we can follow the proof in [CHK07, Theorem 4.3] and proceed to estimate (3.2.2). Notice that the spatial homogeneity of the Delone set in the sense that points do not accumulate neither are too far away, so the sums over indexes of elements of D preserves the properties of the sums over indexes of elements of the lattice $\mathbb{Z}^2$ as the original proofs.

a. Estimate on $\mathbb{E}\{\mathrm{tr}\, P_{\omega,L}(J)\Pi^{\perp}_{n,L}\}$.

The analysis in [CHK07, Eq. 2.6 - 2.10] for the n-th Landau band remains valid taking, for the constants defined therein, $M = 1$ and the operator K defined by

$$K \equiv \left(\frac{H_{B,L}+1}{H_{B,L}-E_m}\right)^2, \qquad \|K\| \leq K_n \equiv \left(1 + \frac{1+J_+}{d_n}\right)^2, \tag{3.2.17}$$

where E_m is an eigenvalue of $H_{B,\lambda,\omega,L}$, $d_n \equiv \min\{\mathrm{dist}(I,B_{n-1}), \mathrm{dist}(I,B_{n+1}\}$ and $J = [J_-, J_+]$.

Then we can obtain the analog of [CHK07, Eq. 4.4],

$$\mathrm{tr}\, P_{\omega,L}(J)\Pi^{\perp}_{n,L} \leq K_n\lambda^2 \max\{m_0, M_0\}^2 \sum_{i,j\in\tilde{\Lambda}} |\mathrm{tr}\;\, u_j P_{\omega,L}(J) u_i K_{ij}|, \tag{3.2.18}$$

where $K_{ij} \equiv \chi_i(H_{B,L}+1)^{-2}\chi_j$, for $\chi \geq 0$ a smooth function of compact support slightly larger than the support of u such that $\chi u = u$. Note that due to the spatial homogeneity of D and the fact that supp u is contained in a cube of side r, the translated supports of u do not overlap.

Now, denote by $\tilde{\Lambda}_0 = \{i,j \in \tilde{\Lambda}/\chi_i\chi_j = 0\}$ and by $\tilde{\Lambda}_0^c = \{i,j \in \tilde{\Lambda}/\chi_i\chi_j \neq 0\}$. For $i,j \in \tilde{\Lambda}_0$, the operator K_{ij} is trace class [BGKS05, Lemma 2.2], [CHK07, Lemma 5.1] and it satisfies the Combes-Thomas estimate,

$$\|K_{ij}\|_1 = \|\chi_i(H_{B,L}+1)^{-2}\chi_j\|_1 \leq C_0' e^{-\tilde{C}_0\|i-j\|}, \tag{3.2.19}$$

where C_0' and $\tilde{C}_0$ are positive constants. So we can use Lemma 3.2.5 to obtain

$$\mathbb{E}\{|\sum_{i,j\in\tilde{\Lambda}_0} \text{tr } u_j P_{\omega,L}(J) u_i K_{ij}|\} \leq \mathbb{E}\{\sum_{i,j\in\tilde{\Lambda}_0} |\text{tr } u_j P_{\omega,L}(J) u_i K_{ij}|\} \tag{3.2.20}$$

$$\leq C_0 8 s(|J|) \sum_{i,j\in\tilde{\Lambda}_0} e^{-\tilde{C}_0\|i-j\|} \tag{3.2.21}$$

$$\leq C_1 s(|J|)|\Lambda|. \tag{3.2.22}$$

where C_1 also depends on r, since $\sharp(\tilde{\Lambda}_L) \leq C_{r,d} L^d$ for $L > R$, see Eq. (3.3.16).

On the other hand, for $i,j \in \tilde{\Lambda}_0^c$, K_{ij} is also trace class [BGKS05, Lemma 2.2] so we can apply Lemma 3.2.5 again, obtaining

$$\mathbb{E}\{\text{tr}\, P_{\omega,L}(J)\Pi_{n,L}^{\perp}\} \leq \; C_2 s(|J|)|\Lambda|, \tag{3.2.23}$$

where $C_2 > 0$ depends on u, I, λ, r and $M = \max\{m_0, M_0\}$.

b. Estimate on $\mathbb{E}\{\text{tr}\, P_{\omega,L}(J)\Pi_{n,L}\}$.

We use the spectral projector $\Pi_{n,L}$ in order to control the trace. Here the key ingredient is the positivity estimate (3.2.3) and the fact that, under our hypotheses on u, there exists a finite constant C_u, depending on u only, such that

$$0 < \tilde{V}_L^2 \leq C_u \tilde{V}_L.$$

Now,

$$\text{tr}\, P_{\omega,L}(J)\Pi_{n,L} \leq \frac{1}{C_n(B,u,R)} \text{tr}\, P_{\omega,L}(J)\Pi_{n,L}\tilde{V}_L\Pi_{n,L} \tag{3.2.24}$$

$$\leq \frac{1}{C_n(B,u,R)} \left\{ \text{tr}\, P_{\omega,L}(J)\tilde{V}_L\Pi_{n,L} - \text{tr}\, P_{\omega,L}(J)\Pi_{n,L}^{\perp}\tilde{V}_L\Pi_{n,L} \right\}. \tag{3.2.25}$$

Then we can proceed as in parts (2) and (3) of the proof of [CHK07, Theorem 4.3], and we finally arrive to the desired result,

$$\mathbb{E}\{\text{tr}\, P_{\omega,L}(J)\} \leq Q_W s(|J|)|\Lambda|. \tag{3.2.26}$$

where the constant $Q_W > 0$ depends on $B, u, R, r, I, \lambda, M$ and n. □

3.3 Proof of Theorem 3.1.4

3.3.1 Dynamical localization in Landau bands

In this section we prove the following

Theorem 3.3.1. *Let H_ω be as before. For any $n = 0, 1, 2, ...$ there exist finite positive constants $\mathbf{B}(n)$ and $K_n(\lambda)$ depending only on n, M, u and ρ such that for all $B \geq \mathbf{B}(n)$ we can perform MSA in the intervals*

$$\Sigma_{B,n,\lambda,\omega} = \sigma_{B,\lambda,\omega} \cap \{E \in \mathcal{B}_n : \ |E - B_n| \geq K_n(\lambda)\frac{\log B}{B}\}, \tag{3.3.1}$$

We have strong HS-dynamical localization at energy levels up to a distance $K_n(\lambda)\frac{\log B}{B}$ from the Landau levels for large B.

For the proof we need to verify the conditions to start the modified Multiscale Analysis, Theorem 2.1.3. As mentioned in the proof of Theorem 2.1.3, this model satisfies properties IAD, R, EDI, SLI and NE. What is left to prove is the existence of a suitable length scale L_0 that satisfies (2.1.5) and UWE. The latter comes from the following improvement in the Wegner estimate of the previous section and it follows [CH96, Theorem 3.1].

Theorem 3.3.2. *There exists $\tilde{B} > 0$ and a constant $Q_n = \tilde{Q}_{n,\lambda,u}\|\rho\|_\infty$ such that for all $B > \tilde{B}$ and for any closed interval $I \subset \mathcal{B}_n \setminus \sigma(H_B)$*

$$\mathbb{E}\{\mathrm{tr}\, P_{B,\lambda,\omega,x,L}(I)\} \leq Q_n \frac{B}{2(dist(I, B_n))^2}|I|L^2. \tag{3.3.2}$$

In particular, for $E_0 \notin \sigma(H_B)$ and all $0 < \epsilon < |E_0 - B_n|$,

$$\mathbb{P}\{\mathrm{dist}(\sigma(H_{B,\lambda,\omega,x,L}), E_0) \leq \epsilon\} \leq Q_n \frac{B}{(|E_0 - B_n| - \epsilon)^2}\epsilon L^2. \tag{3.3.3}$$

Proof. Without loss of generality we work within the first Landau band $\mathcal{B}_0$, containing the Landau level B_0. Set $M = \|V_\omega\|_\infty = \max\{m_0, M_0\}$. Let I be an interval such that $I \subset \mathcal{B}_0 \setminus \{B_0\}$ and $\inf I > B$, so $\mathrm{dist}(I, B_0) > 0$.

Following the same arguments in [CH96, Eq. 3.4 - 3.11], we get

$$\mathbb{E}\{\mathrm{tr}\, P_L(I)\} < \mathrm{dist}(I, B_0)^{-2} M^2 \|\rho\|_\infty |I| \sum_{i,j \in D} \|\Pi^{ij}_{0,L}\|_1, \tag{3.3.4}$$

where $P_L(I)$ stands for $P_{B,\lambda,\omega,x,L}(I)$ and we use the notation $A^{ij} = u_i^{1/2} A u_j^{1/2}$ for any bounded operator A.

To evaluate the sum we consider separately the indices i, j for which $\|i - j\| < 4\delta_u$ and those for which $\|i - j\| \geq 4\delta_u$, with δ_u as in (4.1.2).

Let χ_{ij} be the characteristic function of $\operatorname{supp}(u_i + u_j)$. Again, as in Theorem 3.2.1, the translated supports of u behave in a similar way as in the lattice. Then we follow the same arguments therein and obtain, using [CH96, Lemma 2.1],

$$\sum_{|i-j|<4\delta_u} \|\Pi_{0,L}^{ij}\|_1 \leq \|u\|_\infty^2 \sum_{|i-j|<4\delta_u} \|\chi_{ij}\Pi_{0,L}\chi_{ij}\|_1 \leq C_0 B|\Lambda||\text{supp } u|, \tag{3.3.5}$$

where the constant C_0 actually depends on the index n of the Landau level, which in this case is 0.

Define χ_{ij}^+ to be the characteristic function of the set $\{x \in \mathbb{R}^2 : \|x-i\| < \|x-j\|\}$ and denote $\chi_{ij}^- = 1 - \chi_{ij}^+$. Then we obtain

$$\|\Pi_{0,L}^{ij}\|_1 \leq \|u_j^{1/2}\Pi_{0,L}\chi_{ij}^+\|_2\|\chi_{ij}^+\Pi_{0,L}u_i^{1/2}\|_2 + \|u_j^{1/2}\Pi_{0,L}\chi_{ij}^-\|_2\|\chi_{ij}^-\Pi_{0,L}u_i^{1/2}\|_2.$$

Now, if $|i-j| \geq 4\delta_u$, condition (4.1.2) implies that

$$\text{dist}(\text{supp } \chi_{ij}^+, \text{supp } u_j) \geq \frac{\|i-j\|}{2} - \delta_u \geq k\|i-j\|$$

for some $k > 0$. Similarly for dist(supp χ_{ij}^-, supp u_i). We then obtain

$$\sum_{|i-j|\geq 4\delta_u} \|\Pi_{0,L}^{ij}\|_1 \leq C_1|\text{supp } u||\Lambda|. \tag{3.3.6}$$

Combining (3.3.4), (3.3.5) and (3.3.6) we obtain

$$\mathbb{E}\{\text{tr}\, P_L(I)\} \leq Q_0(\text{dist}(I, B_0))^{-2}\|\rho\|_\infty \epsilon B|\Lambda|,$$

where the constant Q_0 depends on λ, M,$\|u\|_\infty$ and $\operatorname{supp} u$. Taking $I = [E_0 - \epsilon, E + \epsilon]$ for small $\epsilon > 0$ and applying Chebyshev's inequality we obtain (3.3.3).

□

As for the initial length scale estimate (2.1.5) to start the multiscale analysis, we need to verify that for some $L_0 \in 6\mathbb{N}$ sufficiently large (as specified in [GK03]), given $\theta > 0$, $E \in \mathbb{R} \setminus \sigma(H_{B,L})$,

$$\mathbb{P}\left\{\|\Gamma_{x,L_0} R_{B,\omega,x,L_0}(E)\chi_{x,L_0/3}\| \leq \frac{1}{L_0^\theta}\right\} > 1 - \frac{1}{L_0^p}, \tag{3.3.7}$$

for a suitable choice of p, where $\Gamma_{x,L} = \chi_{\bar{\Lambda}_{L-1}(x)\setminus\Lambda_{L-3}(x)}$.

To do so we follow the approach [CH96] to obtain estimates that we will later state as in [GK03]. We need to show that in the annular region between a box of side $L/3$ and L, there exists a closed, connected ribbon where the potential V satisfies the condition $|V(x)+B_n-E| > a > 0$, for $E \neq B_n$ with a good probability ([CH96, Eq. 4.2]). To prove this, Combes and Hislop used bond percolation theory, defining occupied bonds of the lattice as those bonds where the potential satisfies this property. However, in our case there is no need to use percolation theory since this fact is assured by the assumption *(uc)* on the single-site potential. More precisely, we will show that there exist ribbons where the potential is zero almost surely.

Let us consider the *Voronoi diagram* associated to D [OBSC]. Since $\tilde{\Lambda}_L = D \cap \Lambda_L$ is a discrete bounded set, we can write $\tilde{\Lambda}_L = \{p_1, ..., p_n\}$, $n \in \mathbb{N}$. For each site p_i we consider its *Voronoi cell*, defined as

$$\mathcal{V}(p_i) = \{x \in \mathbb{R}^2 : \|x - p_i\| \leq \|x - p_j\|, j \neq i, 1 \leq j \leq n\},$$

i.e., the set of points that are closer to p_i than to any other site in $\tilde{\Lambda}_L$. The Voronoi diagram associated to $\tilde{\Lambda}_L$, denoted by $\mathcal{V}or(\tilde{\Lambda}_L)$ is a subdivision of Λ_L into Voronoi cells,

$$\mathcal{V}or(\tilde{\Lambda}_L) = \bigcup_{1 \leq i \leq n} \mathcal{V}(p_i).$$

The edges and vertices of $\mathcal{V}or(\tilde{\Lambda}_L)$ are polygonal connected lines with the property that the minimal and maximal distances from any site p_i to an edge or vertex are $r/4$ and $\sqrt{d}R/2$, respectively.

Now, take a covering of $\Lambda_{L/3}$ by a finite collection of Voronoi cells, $\mathcal{V}_\Lambda$, which is a convex polygon. Its perimeter is a polygonal line $\mathcal{C}$ that encloses $\Lambda_{L/3}$ such that $\mathcal{C} \cap D = \emptyset$. Taking L big enough with respect to R we have $\mathcal{C} \subset \Lambda_{L-3} \setminus \Lambda_{L/3}$. Moreover, assumption *(uc)* implies that we can always find a ribbon $\mathcal{R}$ associated to $\mathcal{C}$, i.e., a set

$$\mathcal{R} = \{x \in \mathbb{R}^2 : \ dist(x, \mathcal{C}) < \frac{r}{4} - \frac{r}{10}\},$$

such that $V(x) = 0$ for all $x \in \mathcal{R}$ (see Fig. 3.1)

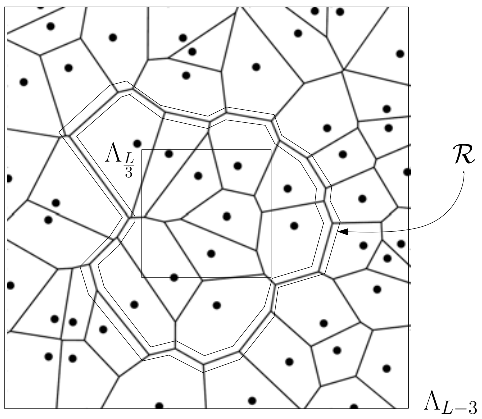

Figure 3.1: Ribbon $\mathcal{R}$ in the Voronoi diagram associated to D. Points represent the support of the Delone–Anderson potential.

Then, condition [CH96, Eq. (4.2)] holds almost surely, therefore [CH96, Corollary 4.1] holds almost surely, and this implies (see [CH96, Proposition 5.1], [GK03, Theorem 4.3])

Theorem 3.3.3. *Let $E = B_n \pm 2a$ for some $n = 0,1,2...$ with $0 < 2a < B$. There exists constants $Y_n, \beta_n > 0$ depending only on n, M, u, δ_u such that for any $0 < \epsilon \leq a$, $L \in 6\mathbb{N}$ and Q_n as in the previous theorem,*

$$\mathbb{P}\left\{\|\Gamma_{x,L} R_{B,\omega,x,L}(E)\chi_{x,L/3}\| \leq Y_n \frac{B}{a\epsilon^2} e^{-\beta_n \min\{aB,\sqrt{B}\}}\right\} > 1 - Q_n \frac{B\epsilon}{a^2} L^2. \tag{3.3.8}$$

Therefore, to satisfy (3.3.7) we need only to verify the conditions

$$Y_n \frac{B}{a\epsilon^2} e^{-\beta_n \min\{aB,\sqrt{B}\}} \leq \frac{1}{L_0^\theta}, \tag{3.3.9}$$

$$Q_n \frac{B\epsilon}{a^2} L_0^2 \leq \frac{1}{L_0^p}, \tag{3.3.10}$$

which can be done in the same way as in the proof of [GK03, Theorem 4.1], yielding Theorem 3.3.1 .

3.3.2 Dynamical delocalization in Landau bands

Theorem 3.3.4. *Under the disjoint bands condition* (3.1.8) *the random Landau Hamiltonian $H_{B,\lambda,\omega}$ exhibits dynamical delocalization in each Landau band $\mathcal{B}_n(B,\lambda)$, i.e. for all $n = 1,2,...$,*

$$\Xi^{DD} \cap \sigma_{B,\lambda,\omega} \cap \mathcal{B}_n(B,\lambda) \neq \emptyset. \tag{3.3.11}$$

In particular, there exists at least one energy $E_{n,\omega}(B,\lambda) \in \mathcal{B}_n(B,\lambda)$ such that for every $\mathcal{X} \in C^\infty_{c,+}(\mathbb{R})$ with $\mathcal{X} \equiv 1$ on some open interval $J \ni E_{n,\omega}(B,\lambda)$ and $p > 0$, we have

$$\mathcal{M}_{B,\lambda}(p,\mathcal{X},T) \geq C_{p,\mathcal{X}} T^{\frac{p}{4}-6}, \tag{3.3.12}$$

for all $T \geq 0$ with $C_{p,\mathcal{X}} > 0$.

This is a consequence of the quantization of the Hall conductance in each Landau band and the fact that in regions of dynamical localization, the Hall conductance is constant, as proven in [GKS07, Section 3]. We recall the main lines of their strategy.

Consider the switch function $h(t) = \chi_{[\frac{1}{2},\infty)}(t)$ and let h_j denote the multiplication by the function $h(x_j)$, $j = 1,2$. The Hall conductance is defined as

$$\sigma_{H_\omega}(B,\lambda,E) = -2\pi i \Theta(P_{B,\lambda,\omega,E}) := \operatorname{tr}\{P_{B,\lambda,\omega,E}[[P_{B,\lambda,\omega,E}, h_1],[P_{B,\lambda,\omega,E}, h_2]]\}, \tag{3.3.13}$$

where $P_{B,\lambda,\omega,E} := P_{B,\lambda,\omega}((-\infty,E])$.

Following the proof of [GKS07, Lemma 3.2] we see that the Hall conductance is constant in connected components of the dynamical localization region, where property SUDEC is valid, as consequence of Theorem 2.1.3. On the other hand, it is well known that for $\lambda = 0$, $\sigma_{H_\omega}(B,\lambda,E) =$

n if $E \in (B_n, B_{n+1})$ for all $n = 0, 1, 2, ...$ Under the disjoint bands condition (3.1.8), if $E \in \mathcal{G}_n(B, \lambda_*)$ for λ_* and some $n \in \{0, 1, 2, ...\}$, we can find some $\lambda_E > \lambda_*$ such that $E \in \mathcal{G}_n(B, \lambda)$ for all $\lambda \in [0, \lambda_E]$. That is, the spectral gaps stay open as λ increases. Then we prove along the lines of [GKS07, Lemma 3.3] that $\sigma_{H_\omega}(B, \lambda, E) = n$ if $E \in \mathcal{G}_n(B, \lambda)$, for all $[0, \lambda_E]$. As the spectral gaps $\mathcal{G}_n(B, \lambda)$ are by definition part of the localization region, this implies that the Hall conductance has the same value in different gaps, which is a contradiction. Therefore, we must have $\Xi^{DD} \cap \sigma_{B,\lambda,\omega} \cap \mathcal{B}_n(B, \lambda) \neq \emptyset$ for every $\omega \in \Omega$.

By Theorems 3.3.1 and 3.3.4 we conclude that there exists a dynamical transition energy in each Landau band as stated in Theorem 3.3.4.

3.3.3 Almost sure existence of spectrum near band edges

Since we deal with a non ergodic random operator, previous results on the nature of the spectrum do not hold in this setting. In particular, we cannot use the characterization of the spectra as a union of spectra of periodic operators as in [GKS07]. We need a more constructive approach and thus, to go back to the argument used in [CH96]. We extend [CH96, Theorem 7.1] to a Delone–Anderson potential to make sure that, although the spectrum $\sigma_{B,\lambda,\omega}$ is random, there exists almost surely some part of $\sigma_{B,\lambda,\omega}$ in the region were we can prove dynamical localization, that is, in the spectral band edges.

Consider the operator acting on $L^2(\mathbb{R}^2)$, $H^D_\omega = H_B + \lambda V^D_\omega$ where $\lambda > 0$ and V^D_ω is defined as in (3.1.2). Recall that

$$V^D_\omega(x) = \sum_{\gamma \in D} \omega_\gamma u_\gamma, \tag{3.3.14}$$

where D is an (r, R)-Delone set, the random variables ω_γ are i.i.d. with absolute continuous probability distribution μ, supp $\mu = [-M, M]$ and $u_\gamma = u(x - \gamma)$. Assume moreover $u \in \mathcal{C}^2$,$\|u\|_\infty = 1$, supp $u \subset \Lambda_r(0)$ and $u(0) = 1$.

Theorem 3.3.5. *Under the disjoint bands conditions, for a random Landau Hamiltonian as stated before and any $n = 0, 1, 2, ...$ there exists a finite positive constant $B(n)$ depending on n, M, u, λ and $K_n(\lambda)$ such that for all $B > B(n)$, the intervals $\Sigma_{B,n,\lambda,\omega}$ in Theorem 3.3.1 are almost surely non empty. More precisely, we prove that there exist finite positive constants C_n, $B(n)$ depending on n, M, u such that for every $B > B(n)$, we have for all $E \in \mathcal{B}_n$,*

$$\sigma(H_\omega) \cap [E - \lambda C_n B^{-1/2}, E + \lambda C_n B^{-1/2}] \neq \emptyset. \tag{3.3.15}$$

For a set $A \in \mathbb{R}^2$ we denote by $\tilde{A}$ the intersection $A \cap D$. Recall that we have, for an arbitrary box $\Lambda_L(x)$ of side $L \in \mathbb{N}$ centered in x:

$$C_{R,d} L^d \leq \sharp(\tilde{\Lambda}_L) = \sharp(D \cap \Lambda_L) \leq C_{r,d} L^d, \tag{3.3.16}$$

where $C_{R,d} = R^{-d}$ and $C_{r,d} = \lceil r^{-d} \rceil$.

Take a sequence $\{x_n\}$ such that $|x_n - x_m| > L$ for every n, m and consider the following sets in the probability space Ω:

$$\Omega^L_\epsilon(x_n) = \{\omega : |\omega_\gamma - \eta| \leq \epsilon \; \forall \gamma \in \tilde{\Lambda}_L(x_n)\}$$

and

$$\Omega_\epsilon^L = \bigcap_N \bigcup_{n \geq N} \Omega_\epsilon^L(x_n), \tag{3.3.17}$$

where $\eta \in [-M, M]$. By the choice of $\{x_n\}$, the events $\Omega_\epsilon^L(x_n)$ and $\Omega_\epsilon^L(x_m)$ are independent for $n \neq m$.

Since the random variables are i.i.d. and (3.3.16) holds for every box $\Lambda_L(x_n)$, we obtain

$$\mathbb{P}\left(\Omega_\epsilon^L(x_n)\right) = \mathbb{P}\left(|\omega_\gamma - \eta| \leq \epsilon, \forall \gamma \in \tilde{\Lambda}_L(x_n)\right) = \mathbb{P}\left(|\omega_\gamma - \eta| \leq \epsilon\right)^{\sharp(D \cap \Lambda_L(x_n))} \tag{3.3.18}$$

$$\geq \mathbb{P}\left(|\omega_\gamma - \eta| \leq \epsilon\right)^{C_{r,d}L^d} \tag{3.3.19}$$

$$= \mu([\eta - \epsilon, \eta + \epsilon])^{C_{r,d}L^d}. \tag{3.3.20}$$

(3.3.21)

Therefore

$$\sum_n \mathbb{P}\left(\Omega_\epsilon^L(x_n)\right) = \infty, \tag{3.3.22}$$

which implies that $\mathbb{P}\left(\Omega_\epsilon^L\right) = 1$, by the Borel–Cantelli lemma.

Given $\delta > 0$, take $\epsilon = \delta/(rL)^d$. We have shown that for $\omega \in \Omega_\epsilon^L$, a set of full measure, there exists an infinite sequence $\{x_n\}$ such that for any $\eta \in [-M, M]$,

$$|\omega_\gamma - \eta| < \frac{\delta}{(rL)^d} \quad \text{for all } \gamma \in \tilde{\Lambda}_L(x_n) \tag{3.3.23}$$

Fix one of these boxes and call it Λ_0 (so Λ_0 depends on ω, but this procedure can be done for all $\omega \in \Omega_0$, the yielding result being uniform in ω).

Without loss of generality, $\tilde{\Lambda}_0$ contains 0. Indeed, if $0 \notin \tilde{\Lambda}_L(x_n)$ for all n, take $L > R$ so that $\tilde{\Lambda}_0 \neq 0$ and take $\gamma_0 \in \tilde{\Lambda}_0$. Consider now the operator

$$H_\omega^{D-\gamma_0} = H_B + \lambda \sum_{\gamma \in D - \gamma_0} \omega_\gamma u_\gamma. \tag{3.3.24}$$

We have that $\sigma(H_\omega^D) = \sigma(H_\omega^{D-\gamma_0})$, since, taking a translation $\tau_{\gamma_0} : \Omega \times D \to \Omega \times (D - \gamma_0)$ defined by $\tau_{\gamma_0}(\omega_\gamma, \gamma) = (\omega_\gamma, \gamma - \gamma_0)$, that associates the same random variable of a point to its translated, we can see H_ω^D is unitarily equivalent to $H_\omega^{D-\gamma_0}$.

Moreover, by what is known for H_ω^D, with full probability there exists a sequence $\{\tilde{x}_n\} = \{x_n - \gamma_0\}$ such that (3.3.23) holds. In particular, since the cube Λ_0 is a cube that satisfies (3.3.23) for H_ω^D, then the cube $\Lambda_{\gamma_0} = \Lambda_0 - \gamma_0$ satisfies (3.3.23) for $H_\omega^{D-\gamma_0}$.

Define

$$V_{\gamma_0}(x) = \eta \sum_{\gamma \in \tilde{\Lambda}_{\gamma_0}} u_\gamma. \tag{3.3.25}$$

Since $\gamma_0 \in \tilde{\Lambda}_0 = \Lambda_0 \cap D$ we have that $0 \in \tilde{\Lambda}_{\gamma_0} = (\Lambda_0 - \gamma_0) \cap (D - \gamma_0)$. Moreover, the assumptions on u, namely that $u(0) = 1$ and the supports of u_γ do not overlap, imply that $V_{\gamma_0}(0) = \eta$. Therefore, without loss of generality we can assume $\tilde{\Lambda}_0$ is centered in 0 and so we work from now on with H_ω^D, V_ω^D and V_0 as in (3.3.25) with $\gamma_0 = 0$.

Remark 3.3.6. *The assumption $u(0) = 1$ is so we can later perform a Taylor expansion around* 0.

Proof of Theorem 3.3.5. From now on L is fixed. For the sake of completeness, we will reproduce the details of [CH96, Appendix 2] with the corresponding adaptations and work in the *0*-th Landau band. Let Π_0 be the Landau projection in the *0*-th Landau band, around the Landau level B_0. Take the normalized function $\phi_0 \in \Pi_0(\mathcal{H})$,defined by

$$\phi_0(x) = \left(\frac{2B}{\pi}\right)^{1/2} e^{-B|x|^2}. \tag{3.3.26}$$

Let $E \in [B_0 - \lambda M, B_0 + \lambda M]$, that is, $E = B_0 + \lambda\eta$ for some $\eta \in [-M, M]$. The case $\eta = 0$ is trivial by the previous Borel–Cantelli argument, as $\{B_n\}_{n\geq 0} \subset \sigma(H_\omega)$ almost surely. Since the argument is analog for $\eta < 0$, in the following we consider only $\eta \in (0, M]$, and write

$$\begin{aligned} \| \left(H_\omega^D - E\right) \phi_0\| &= \| \left(H_\omega^D - B_0 - \lambda\eta\right) \phi_0\| && (3.3.27)\\ &\leq \|\Pi_0(\lambda V_\omega^D - \lambda\eta)\phi_0\| + \lambda\|(1 - \Pi_0)V_\omega^D \phi_0\|. && (3.3.28) \end{aligned}$$

For simplicity we write V_ω instead of V_ω^D. The deterministic result [CH96, Lemma A.1] implies that

$$\lambda\|(1 - \Pi_0)V_\omega\phi_0\| \leq \lambda C_1 B^{-1/2}, \tag{3.3.29}$$

where C_1 is a constant depending only on the single-site potential u. We are left with

$$\begin{aligned} \|\Pi_0(\lambda V_\omega - \lambda\eta)\phi_0\| &\leq \lambda\|(\sum_{\gamma\in\tilde{\Lambda}_0} \omega_\gamma u_\gamma + \sum_{\gamma\in D\setminus\tilde{\Lambda}_0} \omega_\gamma u_\gamma - \eta)\phi_0\| && (3.3.30)\\ &\leq \lambda\|(\sum_{\gamma\in\tilde{\Lambda}_0} \omega_\gamma u_\gamma - \eta)\phi_0\| + \lambda\|\sum_{\gamma\in D\setminus\tilde{\Lambda}_0} \omega_\gamma u_\gamma \phi_0\| && (3.3.31)\\ &\leq \lambda\|(\sum_{\gamma\in\tilde{\Lambda}_0} \omega_\gamma u_\gamma - \eta)\phi_0\| + \lambda M \sum_{\gamma\in D\setminus\tilde{\Lambda}_0} \|u_\gamma\phi_0\|. && (3.3.32) \end{aligned}$$

Recall that

$$\{\gamma \in D : \quad \gamma \in D \setminus \tilde{\Lambda}_0\} \subset \{\gamma \in D : \quad |\gamma| > r\}. \tag{3.3.33}$$

The second term in (3.3.32) can be estimated as in [CH96, Eq. 7.6], where it is shown that

$$\|u_\gamma\phi_0\|^2 = \int_{\mathbb{R}^2} \phi_0(x)^2 u(x-\gamma)^2 dx \leq \|u\|_\infty^2 e^{-2B|\gamma|^2 + 4Br|\gamma|}, \tag{3.3.34}$$

which is summable for γ such that $|\gamma| > r$, yielding that for all $B > B_*$, for a constant B_* big enough,

$$\lambda M \sum_{\gamma \in D \setminus \tilde{\Lambda}_0} \|u_\gamma \phi_0\| \leq \lambda C_2 B^{-1/2}, \tag{3.3.35}$$

where the constant is uniform in B.

As for the first term in (3.3.32), recalling the definition of V_0 from (3.3.25), we write

$$\lambda \|(\sum_{\gamma \in \tilde{\Lambda}_0} \omega_\gamma u_\gamma - \eta)\phi_0\| = \lambda \|(\sum_{\gamma \in \tilde{\Lambda}_0} \omega_\gamma u_\gamma - V_0 + V_0 - \eta)\phi_0\| \tag{3.3.36}$$

$$\leq \lambda \|(\sum_{\gamma \in \tilde{\Lambda}_0} \omega_\gamma u_\gamma - \eta \sum_{\gamma \in \tilde{\Lambda}_0} u_\gamma)\phi_0\| + \lambda \|(V_0 - \eta)\phi_0\| \tag{3.3.37}$$

$$\leq \lambda \| \sum_{\gamma \in \tilde{\Lambda}_0} (\omega_\gamma - \eta) u_\gamma \phi_0\| + \lambda \|(V_0 - \eta)\phi_0\|. \tag{3.3.38}$$

By the choice of Λ_0 the first term in (3.3.38) is

$$\lambda \| \sum_{\gamma \in \tilde{\Lambda}_0} (\omega_\gamma - \eta) u_\gamma \phi_0\| \leq \lambda \delta. \tag{3.3.39}$$

As for the second term in (3.3.38),

$$\|(V_0 - \eta)\phi_0\|^2 = \left(\frac{2}{\pi}\right) \int_{\mathbb{R}^2} |V_0(x) - \eta|^2 e^{-2B\|x\|^2} dx \tag{3.3.40}$$

$$= \left(\frac{2}{\pi}\right) \int_{\mathbb{R}^2} |V_0(B^{-1/2}x) - \eta|^2 e^{-2\|x\|^2} dx. \tag{3.3.41}$$

Now, since $V_0(0) = \eta$, we have

$$|V_0(B^{-1/2}x) - \eta| = |V_0(B^{-1/2}x) - V_0(0)| \tag{3.3.42}$$

and we can perform a Taylor expansion around 0 for V_0, obtaining, since supp $V_0 \subset \Lambda_0$,

$$|V_0(B^{-1/2}x) - V_0(0)| \leq B^{-1/2}\|x\|\|\nabla V_0\|_\infty \leq B^{-1/2} L \|\nabla V_0\|_\infty. \tag{3.3.43}$$

Notice that $\|\nabla V_0\|_\infty \leq C_3$, for a constant C_3 depending only on u, uniformly with respect to $\eta \in [0, M]$. Replacing this in the integral we obtain

$$\|(V_0 - \eta)\phi_0\|^2 = \left(\frac{C_4}{\pi B}\right) \int e^{-2\|x\|^2} dx. \tag{3.3.44}$$

So we obtain once more

$$\lambda \|(V_0 - \eta)\phi_0\| \leq \lambda C_5 B^{-1/2}. \tag{3.3.45}$$

Finally, adding the estimates (A.1),(A.2),(3.3.39) and (3.3.45) yields that for all $B > B_*$,

$$\|H_\omega^D - (B_0 + \lambda\eta)\| \leq \lambda C_5 B^{-1/2} + \delta, \tag{3.3.46}$$

where the bound is uniform in B, $\omega \in \Omega_0$ and in $\eta \in [0, M]$. The same result holds in any Landau band for all B large enough. Therefore, with probability one and for any $E = B_n + \lambda\eta$, we have

$$\sigma(H_\omega^D) \cap [E - \lambda C_5 B^{-1/2} - \delta, E + \lambda C_5 B^{-1/2} + \delta] \neq 0. \tag{3.3.47}$$

Since $\delta > 0$ is arbitrary,

$$\sigma(H_\omega^D) \cap [E - \lambda C_5 B^{-1/2}, E + \lambda C_5 B^{-1/2}] \neq 0, \tag{3.3.48}$$

for every $E \in [B_n, B_n + \lambda M]$. This proves that any gap in the spectrum of H_ω^D in the Landau band cannot exceed a length of order $B^{-1/2}$. □

In particular, since we know by perturbation theory that $\sigma(H_\omega^D) \subset [B_n - \lambda M, B_n + \lambda M]$,we have that for $E = B_n + \lambda M$, that is, in the edge of the Landau band,

$$\sigma(H_\omega^D) \cap [B_n + \lambda M - \lambda C_5 B^{-1/2}, B_n + \lambda M] \neq \emptyset. \tag{3.3.49}$$

On the other hand, by Theorem 3.3.1 we know the localization region is at a distance $K_n(\lambda)\frac{\ln B}{B}$ from the Landau level B_n. If λ is fixed and B is such that

$$K_n(\lambda)\frac{\ln B}{B} < \lambda M - \frac{\lambda C_n}{\sqrt{B}}, \tag{3.3.50}$$

then the region of the spectrum that is almost surely near the band edge, that is above $B_n + \lambda M - \lambda C_n B^{-1/2}$, lies in the localization region, that is above $B_n + K_n(\lambda)\frac{\ln B}{B}$. So we have shown Theorem 3.3.5, that is, for every $n = 0, 1, 2, ...$

$$\Sigma_{B,n,\lambda,\omega} \neq \emptyset \quad \text{for a.e. } \omega \in \Omega. \tag{3.3.51}$$

Chapter 4

Delone–Anderson Operators II: A perturbation of the Laplacian

Contents

4.1 The model

In this chapter, we study a particular non-ergodic model that is a Delone–Anderson perturbation of the Laplacian, or of a perturbation of the Laplacian. In the case where the background Hamiltonian is $-\Delta$ we prove dynamical localization at the bottom of the spectrum using the Bootstrap MSA obtained in Chapter 2 (Section 4.3). Moreover, we obtain an explicit formula for the energy interval where localization holds in terms of the geometric parameters of the Delone set. In section 4.4, we obtain similar results for the case where the background operator is $-\Delta + V_0$, for some bounded potential V_0, using a different method to establish the initial length scale estimate required to apply the MSA. We consider Delone–Anderson perturbations with both regular and singular measures (Delone-Bernoulli type potentials). The Delone–Bernoulli model is studied in section 4.5 using the localization proof by [GK11] *à la* Bourgain–Kenig [BoK05]. Here, we see a distinction between dimensions $d \geq 2$ and $d = 1$, that, we believe, is a consequence of the techniques used to obtain the initial length scale estimate. Part of these results are contained in the article "Dynamical localization for Delone–Anderson operators" [GMRM], in preparation.

Consider the random Schrödinger operator $H_\omega = H_0 + V_\omega$ on $\mathrm{L}^2(\mathbb{R}^d)$ where H_0 is a perturbation of $-\Delta$ and V_ω is a Delone–Anderson potential defined by

$$V_\omega(x) = \sum_{\gamma\in D} \omega_\gamma u(x-\gamma), \tag{4.1.1}$$

satisfying conditions (v1) and (v2), defined in Chapter 3. For the reader's convenience and to make this chapter self contained, we recall,

(v1) The *single-site potential* u is a measurable function such that $\|\sum_{\gamma\in D} u(\cdot-\gamma)\|_\infty = 1$, it has compact support and satisfies

$$u^-\chi_{0,\epsilon_u} \le u \le u^+\chi_{0,\delta_u}, \tag{4.1.2}$$

for some constants $0 < \epsilon_u \le \delta_u < r < \infty$ and $0 < u^- \le u^+ < \infty$.

(v2) $(\omega_\gamma)_{\gamma\in D}$ is a family of i.i.d. random variables, with probability distribution μ of bounded and continuous density ρ such that

$$\rho_+ := \|\rho\|_\infty < \infty, \tag{4.1.3}$$

$$0 \in \operatorname{supp}\rho \subset [-m_0, M_0], \tag{4.1.4}$$

where $0 \le m_0 < \infty$, $0 < M_0 < \infty$. We define the global modulus of continuity of μ as

$$s(\epsilon) = \sup_{E\in\mathbb{R}} \mu([E-\epsilon/2, E+\epsilon/2]) \tag{4.1.5}$$

Since μ is an absolutely continuous probability distribution, $s(\epsilon) \le \rho_+\,\epsilon$.

We denote by V the potential obtained from (4.1.1) by setting all the random variables equal to 1, that is,

$$V(\cdot) = \sum_{\gamma\in D} u(\cdot-\gamma), \tag{4.1.6}$$

For any operator A, we denote by $A_{x,L}$ its restriction to the cube $\Lambda_{x,L}$. For the sake of notation, we will write $A_{x,L}$ or A_L, indistinctively, when no confusion arises.

Remark 4.1.1. The results in this chapter also hold in the case where the random variables ω_γ are independent but not identically distributed. Then, the global modulus of continuity s is defined as $s(\epsilon) = \sup_{\gamma\in D}\sup_{E\in\mathbb{R}} \mu_\gamma([E-\epsilon/2, E+\epsilon/2])$. In such case a disorder assumption is needed for the distributions μ_γ, namely, $s(\epsilon) < \infty$, $\rho_+ = \sup_\gamma \|\rho_\gamma\|_\infty < \infty$ and that for $\epsilon > 0$, there exist constants $C, \tau > 0$ such that $\mu_\gamma([E-\epsilon, E+\epsilon]) \ge C\epsilon^\tau$ for all $\gamma \in D$.

4.2 Uniform Wegner estimates for Delone–Anderson type potentials

Theorem 4.2.1. *i. For $d \ge 1$ let $H_0 = -\Delta$ and $M_0 > 0$. For any $q > 0$, there exists a positive constant $C_{u,d}$ and an energy*

$$E^*(R) = \left(q\frac{C_{u,d}}{R^{d+1}}\right)^2 > 0, \tag{4.2.1}$$

uniform in λ such that for any compact subinterval $I \subset [0, E^*(R)]$ there exists a constant $Q_W = Q_W(\lambda, R, r, u, d, q)$ and a finite scale $\mathcal{L}_* = \mathcal{L}_*(R)$ such that for $L > \mathcal{L}_*$ the following uniform Wegner estimate holds

$$\sup_{x \in \mathbb{R}^d} \mathbb{E}\left\{\operatorname{tr} P_{\lambda,\omega,x,L}(I)\right\} \leq Q_W s(|I|) L^d. \tag{4.2.2}$$

ii. *Let $E_0 \in \mathbb{R} \setminus \sigma(H_0)$ for $H_0 = -\Delta + V_0$, where V_0 is $\mathbb{Z}^d$-periodic. For any bounded interval $I \subset \mathbb{R} \setminus \sigma(H_0)$ there exist a constant $Q_W = Q_W(\lambda, R, r, I, u)$ and a finite scale $\mathcal{L}_*(R)$ such that for every compact subinterval $J \subset I$, (4.2.2) holds.*

iii. *Assume the IDS of H_0 is Hölder continuous with exponent $\delta > 0$ in some open interval I and no further assumption on $s(\epsilon)$. Then there exists a constant $Q'_W = Q'_W(B, \lambda, I, u, R, r, d) > 0$ such that for all compact subintervals $J \subset I$ with $|J|$ small enough, and $0 < \gamma < 1$,*

$$\mathbb{E}\{\operatorname{tr} P_{\lambda,\omega,x,L}(J)\} \leq Q'_W \max\{|J|^{\delta\gamma}, |J|^{-2\gamma} s(|J|)\} L^d. \tag{4.2.3}$$

In particular, if $s(\epsilon) \leq C\epsilon^\zeta$, for some $\zeta \in [0, 1]$, then

$$\mathbb{E}\{\operatorname{tr} P_{\lambda,\omega,x,L}(J)\} \leq Q'_W |J|^{\frac{\zeta\delta}{\delta+2}} L^d. \tag{4.2.4}$$

Proof. For simplicity we omit λ and x from the notation. We follow the proof by [CHK07, CHK03], as done for the Landau Hamiltonian in Chapter 3, that relies in decomposing $\mathbb{E}\{\operatorname{tr} P_{\omega,L}(J)\}$ with respect to the free spectral projector of an interval $\tilde{J}$, such that $J \subset \tilde{J}$ and $d_J = \operatorname{dist}(J, \tilde{J}^c) > 0$, that is

$$\operatorname{tr} P_{\omega,L}(J) = \operatorname{tr} P_{\omega,L}(J) P_{0,L}(\tilde{J}) + \operatorname{tr} P_{\omega,L}(J) P_{0,L}(\tilde{J}^c). \tag{4.2.5}$$

The key step in estimating the first term of the r.h.s is to prove a positivity estimate as in [CHK07, Theorem 2.1]. Once this is established, the proof follows as in [CHK07], since the second term can be estimated in a standard way using Combes-Thomas type estimates (see Proof of Theorem 3.2.1-(a)). For the first term, the key is to obtain a positivity estimate analog to [CHK07, Theorem 2.1], which we do in the following.

To prove (i) we need to show that there exists an energy $E^*(R) > 0$ and a constant $C_{u,d,R} > 0$ such that for every $\tilde{I} \subset [0, E^*(R)]$ we have,

$$P_{0,L}(\tilde{I}) V_L P_{0,L}(\tilde{I}) \geq C_{u,d,R} P_{0,L}(\tilde{I}). \tag{4.2.6}$$

We take advantage of an averaging trick used in [BoK05, GHK07]. We compare the potential $\tilde{V}_{x,L}$ to a spatially averaged potential defined by

$$\bar{V}_L(\cdot) := \frac{1}{R^d} \int_{\Lambda_R(0)} V_L(\cdot - a) da \geq \frac{C_u}{R^d} \chi_{\Lambda_L}(\cdot). \tag{4.2.7}$$

Let $L > R$ and let φ be a normalized eigenfunction in the range of $P_{0,L}(\tilde{I})$, we have

$$\left\langle \varphi, P_{0,L}(\tilde{I}) V_L P_{0,L}(\tilde{I})\varphi \right\rangle = \langle \varphi, V_L \varphi \rangle \tag{4.2.8}$$

$$= \langle \varphi, \bar{V}_L \varphi \rangle + \langle \varphi, (V_L - \bar{V}_L)\varphi \rangle \tag{4.2.9}$$

$$\geq \frac{C_u}{R^d} - \frac{1}{R^d} \int_{\Lambda_R(0)} \langle \varphi(\cdot + a), V_L \varphi(\cdot + a) \rangle - \langle \varphi, V_L \varphi \rangle \, da \tag{4.2.10}$$

$$\geq \frac{C_u}{R^d} - \frac{1}{R^d} \int_{\Lambda_R(0)} | \langle \varphi(\cdot + a), V_L \varphi(\cdot + a) \rangle - \langle \varphi, V_L \varphi \rangle | da \tag{4.2.11}$$

$$= \frac{C_u}{R^d} - \frac{1}{R^d} \int_{\Lambda_R(0)} | \langle \varphi(\cdot + a) - \varphi, V_L \varphi(\cdot + a) \rangle + \langle \varphi, V_L \varphi(\cdot + a) \rangle - \langle \varphi, V_L \varphi \rangle | da \tag{4.2.12}$$

$$= \frac{C_u}{R^d} - \frac{1}{R^d} \int_{\Lambda_R(0)} | \langle \varphi(\cdot + a) - \varphi, V_L \varphi(\cdot + a) \rangle + \langle \varphi, V_L (\varphi(\cdot + a) - \varphi) \rangle | da \tag{4.2.13}$$

$$\geq \frac{C_u}{R^d} - \frac{1}{R^d} \int_{\Lambda_R(0)} 2\|\varphi(\cdot + a) - \varphi\| \|V_L\| da \tag{4.2.14}$$

$$\geq \frac{C_u}{R^d} - \frac{1}{R^d} \int_{\Lambda_R(0)} 2|a| \|\nabla_L \varphi\| da \tag{4.2.15}$$

$$\geq \frac{C_u}{R^d} - 2R\sqrt{d}\xi, \tag{4.2.16}$$

$$\tag{4.2.17}$$

where $\|\nabla_L \varphi\| = \xi$ and we use the fact that $\|\varphi(\cdot + a) - \varphi\| \leq |a| \|\nabla_L \varphi\|_L$. Take

$$\xi = q \frac{C_u}{2\sqrt{d} R^{d+1}} = q \frac{C_{u,d}}{R^{d+1}}, \quad \text{for some } q \in (0,1), \tag{4.2.18}$$

then

$$\frac{C_u}{R^d} - 2R\sqrt{d}\xi = (1-q)\frac{C_u}{R^d} := C_{q,u,d,R} > 0. \tag{4.2.19}$$

Choose

$$E^*(R) = \left(q \frac{C_{u,d}}{R^{d+1}} \right)^2. \tag{4.2.20}$$

Then, since $\xi^2 \leq E^*(R)$, we obtain that for any $q \in (0,1)$ and for any $\varphi \in \text{Ran } P_{0,L}(\tilde{I})$, there exists a constant $C_{q,u,d,R} > 0$ such that for $\tilde{I} \subset [0, E^*(R))$,

$$\left\langle \varphi, P_{0,L}(\tilde{I}) V_L P_{0,L}(\tilde{I})\varphi \right\rangle \geq C_{q,u,d,R} \left\langle \varphi, P_{0,L}(\tilde{I})\varphi \right\rangle. \tag{4.2.21}$$

This gives the crucial estimate to prove an optimal Wegner estimate as in [CHK07, Theorem 1.1].

As for (ii), note that in this case $\operatorname{tr} P_{0,L}(\tilde{J}) = 0$ if $\tilde{J} \subset \mathbb{R} \setminus \sigma(H_0)$, so we only need to estimate the second term in the r.h.s. of (4.2.5), where we do not need the positivity estimate (4.2.6) for $P_{0,L}$. This is done as in the proof of Theorem (3.2.1)–(a).

In case (iii) we can estimate the first term in the r.h.s. of (4.2.5) without using the analog of (4.2.6) for $P_{0,L}$. Instead, the Hölder continuity of the IDS of the non perturbed operator implies that there exists a constant $C > 0$ such that

$$\operatorname{tr} P_{0,L}(\tilde{J}) \leq C|\tilde{J}|^\delta |\Lambda|,$$

and so, for $0 < \gamma < 1$

$$\operatorname{tr} P_{\omega,L}(J) P_{0,L}(\tilde{J}) \leq C|J|^{\gamma\delta}|\Lambda|. \tag{4.2.22}$$

Following the lines of [CHK07] and writing explicitly the dependence on d_J, we have

$$\mathbb{E}\{\operatorname{tr} P_{\omega,L}(J) P_{0,L}(\tilde{J}^c)\} \leq \frac{Q'_W}{d_J^2} s(|J|)|\Lambda|,$$

by taking $d_J = |J|^\gamma$ we obtain the desired result. Furthermore, if $s(\epsilon)$ is ζ-Hölder continuous, we get, taking γ such that $\gamma\delta = \zeta - 2\gamma$,

$$\begin{aligned} \mathbb{E}\{\operatorname{tr} P_{\omega,L}(J)\} &\leq Q'_W \max\{|J|^{\gamma\delta}, |J|^{\zeta-2\gamma}\} L^2 && (4.2.23)\\ &\leq Q'_W |J|^{\frac{\zeta\delta}{\delta+2}} L^2, && (4.2.24) \end{aligned}$$

where Q'_W depends on u, I, λ, R, r and M.

□

4.3 Dynamical localization for a Delone–Anderson perturbation of the Laplacian

Consider a Delone set D of characteristics r, R and its associated Delone–Anderson operator $H_\omega = -\Delta + V_\omega$ on $\mathrm{L}^2(\mathbb{R}^d)$, with V_ω defined in (4.1.1). We assume $m_0 = 0$, so the random variables in the definition of V_ω are supported in the interval $[0, M]$ for some $M > 0$ fixed. Then $\sigma(H_\omega) = [0, \infty)$ for a.e. $\omega \in \Omega$. In this section we prove the following

Theorem 4.3.1. *There exist positive constants C' and A, depending on the parameters of the model d, u, μ, M, and an energy*

$$E_* = \frac{C'}{R^{2(d+1)} \left|\log AR\right|^{2/d}}, \tag{4.3.1}$$

such that for every small open interval $I \subset [0, E_*)$, we have $I \subset \Sigma_{MSA}$. In particular, H_ω exhibits dynamical localization in $[0, E_*)$.

4.3.1 The Initial Length Scale Estimate (ILSE)

Since, by Theorem 4.2.1–(i), H_ω satisfies a uniform Wegner estimate, in order to apply the Bootstrap MSA (Theorem 2.1.3) it remains to prove an initial length scale estimate (ILSE). We do this by showing that there exists a length scale L_0 such that a gap of size E_* occurs above $0 = \inf \sigma(H_\omega)$ in the spectrum of $H_{\omega,L}$ with good enough probability. Applying the Combes-Thomas estimate inside the gap will yield the desired decay of the finite-volume resolvent, and thus proving ILSE (2.1.5), in the interval $[0, E_*)$.

The following is a reformulation of [G08, Proposition 3.1] adapted to our setting.

Proposition 4.3.2. *Fix $p > 0$. There exists a constant $C = C_{u,\mu,d,p,M_0}$ such that for all scales $L \geq 1$ and $x \in \mathbb{R}^d$ we have*

$$\mathbb{P}\left(H_{\omega,x,L} \geq CR^{-2(d+1)}(\ln L)^{-\frac{2}{d}}\right) \geq 1 - \frac{1}{L^{pd}} \tag{4.3.2}$$

Proof. Fix $K > 1$. By assumption (v1) (4.1.2), there exists a constant $C_{u,d}$ such that, uniformly with respect to the center of the box $x \in \mathbb{R}^d$, for some $K > 1$,

$$\bar{V}_{\omega,L}(\cdot) := \frac{1}{(KR)^d} \int_{\Lambda_{KR}(0)} V_{\omega,L}(\cdot - a)da \geq \frac{C_{u,d}}{R^d} Y_{\omega,L}\chi_{\Lambda_L}(\cdot), \tag{4.3.3}$$

where

$$Y_{\omega,L} := \min_{\xi \in \tilde{\Lambda}} \frac{1}{K^d} \sum_{\zeta \in \tilde{\Lambda}_{\frac{K}{3}}(\xi)} \omega_\zeta. \tag{4.3.4}$$

It follows from standard estimates that, if $\bar{\mu}$ is the mean of the probability measure μ, there exists $A_\mu > 0$ such that

$$\mathbb{P}\left(\frac{1}{K^d} \sum_{\zeta \in \tilde{\Lambda}_{\frac{K}{3}}(\xi)} \omega_\zeta \leq \frac{\bar{\mu}}{2}\right) \leq e^{-A_\mu K^d}. \tag{4.3.5}$$

Therefore,

$$\mathbb{P}\left(\bar{V}_{\omega,L} \geq \frac{C_{u,d}\bar{\mu}}{2R^d}\chi_{\Lambda_L}\right) \geq 1 - L^d e^{-A_\mu K^d}, \tag{4.3.6}$$

and in turn, for the operator defined by

$$\bar{H}_{\omega,L} = -\Delta_L + \bar{V}_{\omega,L}, \tag{4.3.7}$$

we have

$$\mathbb{P}\left(\bar{H}_{\omega,L} \geq \frac{C_{u,d}\bar{\mu}}{2R^d}\right) \geq 1 - L^d e^{-A_\mu K^d}. \tag{4.3.8}$$

Take ω in the event whose probability is given by the previous estimate. If $\varphi \in \mathcal{C}_c^\infty(\Lambda_L)$, with $\|\varphi\| = 1$, we can proceed as in the previous section and estimate

$$\begin{aligned}
\langle \varphi, H_{\omega,L}\varphi \rangle &= \langle \varphi, \bar{H}_{\omega,L} \rangle - \langle \varphi, (\bar{V}_{\omega,L} - V_{\omega,L})\varphi \rangle && (4.3.9)\\
&\geq \frac{C_{u,d}\bar{\mu}}{2R^d} - \frac{1}{(KR)^d} \int_{\Lambda_{KR}(0)} \left(\langle \varphi(\cdot + a), V_{\omega,L}\varphi(\cdot + a)\rangle - \langle \varphi, V_{\omega,L}\varphi \rangle\right) da && (4.3.10)\\
&\geq \frac{C_{u,d}\bar{\mu}}{2R^d} - \frac{2}{(KR)^d} \int_{\Lambda_{KR}(0)} \|\varphi(\cdot + a) - \varphi\| \|V_{\omega,L}\|_\infty && (4.3.11)\\
&\geq \frac{C_{u,d}\bar{\mu}}{2R^d} - KRM\sqrt{d}\|\nabla\varphi\|, && (4.3.12)\\
& && (4.3.13)
\end{aligned}$$

where $M = \sup_\omega \|V_{\omega,L}\|_\infty$ and we used that $\|\varphi(\cdot + a) - \varphi\| \leq |a| \, \|\nabla\varphi\|$. Notice that, since $V_{\omega,L} \geq 0$,

$$\|\nabla\varphi\|^2 = \langle\varphi, -\Delta\varphi\rangle \le \langle\varphi, H_{\omega,L}\varphi\rangle. \tag{4.3.14}$$

Then

$$\langle\varphi, H_{\omega,L}\varphi\rangle \ge \frac{C_{u,d}\bar{\mu}}{2R^d} - KRM\sqrt{d}\langle\varphi, H_{\omega,L}\varphi\rangle^{1/2}. \tag{4.3.15}$$

Taking $\varphi \in \text{Rang}\, P_{(-\infty,1]}(H_{\omega,L})$, we have that $\langle\varphi, H_{\omega,L}\varphi\rangle < 1$ and therefore,

$$\langle\varphi, H_{\omega,L}\varphi\rangle \ge \frac{(C_{u,d}\bar{\mu})^2}{dM^2 4^3 R^{2d+2}K^2} := \frac{C_{u,d,\mu,M}}{R^{2d+2}K^2}. \tag{4.3.16}$$

We then have

$$\mathbb{P}\left(H_{\omega,L} \ge C_{u,d,\mu,M}R^{-(2d+2)}K^{-2}\right) \ge 1 - L^d e^{-A_\mu K^d}. \tag{4.3.17}$$

Given $p > 0$, take $K = \left(\frac{(p+1)d}{A_\mu}\ln L\right)^{1/d}$. Notice that $K > 1$ for L big enough depending on A_μ, p, d. Then

$$\mathbb{P}\left(H_{\omega,L} \ge CR^{-(2d+2)}(\ln L)^{-2/d}\right) \ge 1 - L^{pd}, \tag{4.3.18}$$

where $C = C_{u,d,\mu,M}\left(\frac{A_\mu}{(p+1)d}\right)^{1/d}$. □

The Combes-Thomas estimate [S, Theorem 2.4.1] states that for $E \in J$ such that $\eta = \text{dist}(E, \sigma(H_{\omega,x,L})^c \cap I) > 0$, there exist positive constants c_1 and c_2 such that

$$\|\Gamma_{x,L}R_{x,L}(E)\chi_{x,L/3}\| \le \frac{c_1}{\eta}e^{-c_2\eta^{1/2}\,\text{dist}(\text{supp}\,\Gamma_{x,L},\Lambda_{x,L/3})}. \tag{4.3.19}$$

Because of the size of the spectral gap in the spectrum of $H_{\omega,L}$ above 0, it is enough to take $E \in [0, a_L/2]$, where $a_L := CR^{-(2d+2)}(\ln L)^{-2/d}$, so that $\eta = \text{dist}(E, a_L) \ge a_L/2$. In this case the Combes–Thomas estimate gives

$$\|\Gamma_{x,L}R_{x,L}(E)\chi_{x,L/3}\| \le c_3 R^{(2d+2)}(\ln L)^{2/d}e^{-c_4 LR^{-(d+1)}(\ln L)^{-1/d}}, \tag{4.3.20}$$

where we used the fact that $\text{dist}(\text{supp}\,\Gamma_{x,L}, \Lambda_L) \ge L/2$ and

$$c_3 = \frac{2c_1}{C_{u,d}}, \quad c_4 = \frac{c_2\sqrt{C}}{2\sqrt{2}}. \tag{4.3.21}$$

This gives the desired decay of the finite-volume resolvent proving ILSE in the interval $[0, E_*)$, where

$$E_* = a_L/2 = C'R^{-(2d+2)}(\ln L)^{-2/d} \quad \text{for some } C' = C'_{u,d,\mu,M,A_\mu,p} > 0. \tag{4.3.22}$$

4.3.2 An explicit formula for the energy E_*

In order to obtain the dependence of the energy E_* on the parameters of the (r, R)-Delone set, we need to know how big the initial length scale L_0 has to be in terms of these parameters. Let us recall an explicit finite-volume criteria given by [GK03, Theorem 2.5], adapted to our setting,

Theorem 4.3.3. *Let H_ω be a random operator such that Assumptions R, IAD, SLI, EDI, IAD, NE, UWE, and USGEE, defined in Sections 2.1,2.2, hold in an open interval I. Set*

$$\mathcal{L} = \max\left\{3\varrho, 42, 3\left(\frac{107^d}{2}\right)^{\frac{2}{d}}, \frac{1}{37}(16\cdot 60^d Q_W)^{2/d}\right\}, \tag{4.3.23}$$

where ρ is the IAD parameter and Q_W is the constant in the UWE. Suppose that for some $L_0 \geq \mathcal{L}$, $L_0 \in 6\mathbb{N}$ and $E \in \Sigma \cap I$,

$$\mathbb{P}\left(90^d\gamma_I^2(37L_0)^{2d}\|\Gamma_{L_0}R_{\omega,L_0}(E)\chi_{L_0/3}\| < 1\right) \geq 1 - \frac{2}{344^d}, \tag{4.3.24}$$

where γ_I is the constant in the SLI. Then $E \in \Sigma_{MSA}$.

We work in the interval $[0, E_*]$. Clearly from the definition of the random potential, H_ω satisfies property R and IAD with with $\rho < r$ and therefore properties SLI, EDI and USGEE (as seen in Secton 2.2). By Theorem 4.2.1 , H_ω satisfies UWE with a constant Q_W proportional to (4.2.19). Recalling (4.3.20), according to (4.3.23) L_0 has to satisfy

$$L_0 \geq \frac{1}{37}(16\cdot 60^d Q_W)^{2/d} = c_5R^2, \quad c_5 := \frac{C_{u,d}}{37}(16\cdot 60^d)^{2/d}, \tag{4.3.25}$$

$$(37^2 90)^d\gamma_I^2 L_0^{2d}c_3R^{(2d+2)}(\log L_0)^{2/d}e^{-c_4L_0R^{-(d+1)}(\ln L_0)^{-1/d}} < 1. \tag{4.3.26}$$

Choose

$$L_0 = A^{d+2}R^{(d+2)\mathrm{sgn}(R-1)}, \tag{4.3.27}$$

where $A \geq 1$ is a constant to be chosen later.

For the moment, assume $R \geq 1$. Then, for L_0 to satisfy (4.3.25), we need to have $A^{d+2} \geq c_5$. On the other hand, to satisfy (4.3.26), we need to have

$$c_6(AR)^{2d(d+2)}R^{2d+2}((d+2)\log AR)^{2/d}\exp\left(-\frac{c_4A^{d+1}AR}{(d+2)^{1/d}(\log AR)^{\frac{1}{d}}}\right) < 1, \tag{4.3.28}$$

where $c_6 = (37^2 90)^d\gamma_I^2 c_3$. Since $AR \geq 1$ we always have $\log AR \leq (AR)^{\frac{d}{2}}$, so the condition is satisfied if

$$c_7(AR)^{2d(d+2)+1}R^{2d+2} < e^{c_8A^{d+1}\sqrt{AR}}, \tag{4.3.29}$$

where $c_7 = c_6(d+2)^{2/d}$, $c_8 = c_4/(d+2)^{1/d}$ Since $A \geq 1$, (4.3.29) will hold if

$$c_7(AR)^{2d^2+6d+3} < e^{c_8\sqrt{AR}}. \tag{4.3.30}$$

It is enough to ask for A to satisfy

$$c_7 A^{2d^2+6d+4} < e^{c_4\sqrt{A}}, \tag{4.3.31}$$

since the same will hold for any number larger than A, in particular, for AR. As $A \geq 1$ and $\sqrt{AR} \leq A^{d+1}\sqrt{AR}$, then (4.3.29), and therefore (4.3.26), follows.

If $R < 1$, define $\tilde{R} = \frac{1}{R}$. Then $L_0 = (A\tilde{R})^{d+2}$ where $\tilde{R} > 1$ and the same estimates as before hold for $A \geq 1$ big enough uniformly in $\tilde{R}$ satisfying (4.3.31).

This proves that we can pick a positive constant A big enough, uniform in R, depending on d, γ_I, u, c_1, such that the conditions for Theorem 2.1.3 hold for $E \in [0, E_*)$. Since $L_0 = A^{d+2}R^{(d+2)\operatorname{sgn}(R-1)}$, from (4.3.22) we have

$$E_* = \frac{C'}{R^{2(d+1)}\,|\log AR|^{2/d}} \quad \text{with } C' = C'_{u,d,\mu,M,A_\mu,p}, A = A_{u,d,\gamma_I,c_1} > 0. \tag{4.3.32}$$

4.4 Dynamical localization for a perturbation of the Laplacian with a Delone–Anderson potential

Consider a Delone set D of characteristics r, R, and the operator $H_\omega = H_0 + V_\omega$ $\mathrm{L}^2(\mathbb{R}^d)$ where $H_0 = -\Delta + V_0$ with V_0 a bounded potential. The random potential V_ω is defined by

$$V_\omega(x) = \sum_{\gamma\in D} \omega_\gamma u(x-\gamma), \tag{4.4.1}$$

satisfying properties (v1) and

(v3) $(\omega_\gamma)_{\gamma\in D}$ is a family of i.i.d. random variables, with common probability distribution μ such that

$$\operatorname{supp}\mu = [0, M], \quad \text{for some } M \geq 1, \tag{4.4.2}$$

$$\mu[0,t] \leq ct^\tau, \quad \text{for small } t, \text{ a positive constant } c \text{ and } \tau > d/2. \tag{4.4.3}$$

Then $E_0 = \inf\sigma(H_0) = \inf\sigma(H_\omega)$ a.e. $\omega \in \Omega$. We have the following result for the operator H_ω,

Theorem 4.4.1. *Let $\beta \in (d/(2\tau), 1)$ be fixed. Then, there exist positive constants c_1, c_2 and A, depending on d, M, I, u, V_0, β, and an energy E_* given by*

$$E_* = E_0 + c_3 R^{-c_4\,R^{4/3}\,sgn(R-1)} A^{-c_4\,A^{4/3}}, \tag{4.4.4}$$

such that for every small open interval $I \subset [E_0, E_*)$, we have $I \subset \Sigma_{MSA}$. In particular, H_ω exhibits dynamical localization on $[E_0, E_*)$.

Let $H_{\omega,L}$ and $H_{0,L}$ denote the restriction of H_ω and H_0 to the cube $\Lambda_L(x)$ with Dirichlet boundary conditions. We write

$$\lambda_L(t) = \inf\sigma(H_{t,L}), \quad \text{where } H_{t,L} = H_{0,L} + tV_L, \tag{4.4.5}$$

where V_L, we recall, denotes the restriction of the potential V obtained from V_ω by setting $\omega_\gamma = 1$ for all $\gamma \in D$. We use the unique continuation principle for the operator $H_{0,L}$, with the Delone-type perturbation V_L. By [RMV12] (see Appendix A.1, A.3), we know that there exists a constant $C_{UCP} > 0$ independent on the scale L, such that for all $t \in [0,1]$, there exists a constant $\kappa = u_- C_{UCP}$ such that

$$\lambda_L(t) \geq \lambda_L(0) + \kappa t \geq E_0 + \kappa t, \tag{4.4.6}$$

where u_- comes from (4.1.2) and C_{UCP} depends on the parameter R of the Delone set D.

Since V_0 is bounded, it satisfies an optimal Wegner estimate at the bottom of the spectrum as proved in [RMV12].

In order to prove dynamical localization through Theorem 2.1.3, it remains to show the existence of an initial length scale estimate (2.1.5). As done previously, we aim to show the existence of a gap in the spectrum of $H_{\omega,L}$ above E_0 with good probability, such that we can apply Combes-Thomas estimate inside the gap and obtain ILSE near E_0. We proceed as in [KSS98, Section 4].

$$\mathbb{P}\left(V_{\omega,L} > tV\right) = \mathbb{P}\left(\omega_\gamma > t, \quad \forall \gamma \in \Lambda_{x,L} \cap D_2\right) = 1 - \mathbb{P}\left(\exists \text{ at least one } \gamma \in \Lambda_{x,L} \cap D_2 \, : \omega_\gamma \leq t\right) \tag{4.4.7}$$

$$\geq 1 - \left|\Lambda_{x,L} \cap D_2\right| \mu[0,t] \tag{4.4.8}$$

$$\geq 1 - C_{r_2,d} L^d c t^\tau, \tag{4.4.9}$$

where we used the fact $|\Lambda_{x,L} \cap D_2| \leq C_{r_2,d} L^d$ uniformly in $x \in \mathbb{R}^d$. Now pick

$$t = \frac{1}{\kappa L^{2\beta}} \quad \text{for some } \beta \in (0,1). \tag{4.4.10}$$

Note that for L big enough depending on u, C_{UCP}, β, we have $t \leq 1$. Then

$$\mathbb{P}\left(\omega_\gamma > t, \quad \forall \gamma \in \Lambda_L\right) \geq 1 - \frac{C_{r_2,d} c}{\kappa^\tau L^{2\beta\tau - d}}. \tag{4.4.11}$$

Since $\tau > d/2$, we can pick β such that $\frac{d}{2\tau} < \beta < 1$ and therefore $2\beta\tau - d > 0$. This implies

$$\mathbb{P}\left(V_{\omega,L} > tV\right) \geq 1 - \frac{c}{\kappa^\tau L^\xi}, \tag{4.4.12}$$

where $\xi = 2\beta\tau - d$. We denote this event by Ω_0. For any $\omega \in \Omega_0$, since $V_\omega \geq 0$, we have $H_{\omega,L} = H_{0,L} + V_{\omega,L} > H_{0,L} + tV_L$, that is, $\inf \sigma(H_{\omega,L}) \geq \lambda_L(t)$. In particular, by the choice of t,

$$\inf \sigma(H_{\omega,L}) \geq E_0 + \frac{1}{L^{2\beta}}. \tag{4.4.13}$$

We have shown that with a probability given by (4.4.12), there is a gap of size $1/L^{2\beta}$ above E_0 in the spectrum of $H_{\omega,L}$.

Now, consider the interval $[E_0, E_0 + a_L/2]$, where

$$a_L := \frac{1}{L^{2\beta}}. \tag{4.4.14}$$

By the Combes-Thomas estimate (4.3.19), there exists constants c_1 and c_2 such that for any $E \in [E_0, E_0 + a_L/2]$

$$\|\Gamma_{x,L} R_{x,L}(E) \chi_{x,L/3}\| \leq c_1 L^{2\beta} e^{-c_2 L^{1-\beta}}. \tag{4.4.15}$$

For L big enough depending on c_1, c_2, β we obtain ILSE at the bottom of the spectrum. Analogously to the previous section, one can estimate how big the initial length scale L needs to be in terms of the parameter of the model in order to start the MSA, using Theorem 4.3.3.

Consider the uniform Wegner estimate given by [RMV12, Theorem 4.11] for $H_{\omega,L}$ restricted to the box $\Lambda_L(x)$ with Dirichlet boundary conditions (the Wegner estimate given by Theorem 4.2.1-(a) can be analyzed in an analogous way). The quantitative unique continuation principle (see Appendix A.1, A.3) implies the following positivity estimate: for any open interval $I \subset [E_0, E_0 + \kappa/2]$, where $\kappa = u_- C_{UCP}$,

$$P_{0,L}(I)\, V\, P_{0,L}(I) \geq \frac{1}{2} \kappa P_{0,L}(I). \tag{4.4.16}$$

This enters in the constant Q_W of the Wegner estimate as

$$Q_W = \frac{C}{\kappa^2}, \tag{4.4.17}$$

for some constant C depending on d, M, I (see (3.2.3), (3.2.24) in the proof of Theorem 3.2.1-(a), or the proof of [CHK07, Theorem 4.1]). From Appendix A.1, we see that the constant C_{UCP} depends on the parameter R and, in consequence,

$$Q_W = C'\, R^{\zeta\, R^{4/3}}, \tag{4.4.18}$$

for some positive constants C' depending on d, M, u, I, and ζ on V_0, I, d.

Now, according to Theorem 4.3.3 and condition (4.4.10), the initial length scale estimate L_0 must satisfy

$$L_0 \geq (\alpha_1 Q_W)^{2/d} = (\alpha_2\, R^{\zeta R^{4/3}})^{2/d}, \quad \text{and } L_0 \geq (u_- R^{\zeta\, R^{4/3}})^{1/(2\beta)}, \tag{4.4.19}$$

where α_1, α_2 depend on d, M, u, I.

Take

$$L_0 = \alpha_3 R^{\frac{2}{(d/2+2\beta)} \zeta\, R^{4/3} \mathrm{sgn}(R-1)} A^{\frac{2}{(d/2+2\beta)} \zeta\, A^{4/3}}, \tag{4.4.20}$$

for some constant $A > 0$ to be fixed. Let us write $\alpha_5 = \frac{2}{(d/2+2\beta)} \zeta$. Without loss of generality, consider the case $R \geq 1$. moreover, from (4.3.24) and (4.4.15), L_0 has to satisfy

$$\alpha_6 L_0^{2d+2\beta} < e^{c_2 L_0^{1-\beta}}, \tag{4.4.21}$$

where $\alpha_6 = c_1 (37^2 90)^d \gamma_I^2$. Take A big enough, depending on d, β, d, M, u, I, such that

$$\alpha_6 A^{(2d+2\beta)\alpha_5 A^{4/3}} < \exp\left(c_2 A^{c_2 e^{\alpha_5 (1-\beta) A^{4/3}}} \right). \tag{4.4.22}$$

Since this inequality also holds for any number greater than A, in particular for AR, we get (4.4.21).

Therefore, there exist positive constants c_3, c_4 and A, depending on d, M, I, u, V_0, β, and an energy E_* given by

$$E_* = E_0 + c_3 R^{-c_4 R^{4/3} \operatorname{sgn}(R-1)} A^{-c_4 A^{4/3}}, \tag{4.4.23}$$

such that in any small open interval $I \subset [E_0, E_*)$, we can verify the hypotheses of Theorem 2.1.3, and therefore $I \subset \Sigma_{MSA}$.

4.5 Dynamical localization for a Delone-Bernoulli model

Consider a (r, R)-Delone set D and its associated Delone–Anderson operator $H_\omega = -\Delta + V_0 + V_\omega$ on $\mathrm{L}^2(\mathbb{R}^d)$. The background potential V_0 is bounded and V_ω is a Delone-Bernoulli potential, defined by (4.4.1), satisfying properties (v1) (4.1.2) and

(v4) $(\omega_\gamma)_{\gamma \in D}$ is a family of i.i.d. Bernoulli random variables, with common probability distribution μ such that

$$\operatorname{supp} \mu = \{0, 1\}, \tag{4.5.1}$$

$$\mathbb{P}(\omega_0 = 0) = \beta, \quad \mathbb{P}(\omega_0 = 1) = 1 - \beta, \quad \text{for some } \beta \in (0, 1). \tag{4.5.2}$$

We denote by $E_0 = \inf \sigma(H_0) = \inf \sigma(H_\omega)$ for a.e. $\omega \in \Omega$.

Theorem 4.5.1.

i. In dimension $d \geq 2$, H_ω exhibits dynamical localization at the bottom of the spectrum.

ii. In dimension $d = 1$, there exists a positive constant C depending on u, V_0 and I such that if

$$\beta < e^{-C R}, \tag{4.5.3}$$

then H_ω exhibits dynamical localization in some interval I, at the bottom of the spectrum.

We can no longer use Theorem 2.1.3 as before since it does not hold for Bernoulli measures, but instead we use results from [GK11] and proceed as done in [G08] for the case $V_0 = 0$. We need to prove that we can find a gap in the spectrum of $H_{\omega,L}$ above E_0 with good probability such that we can apply Combes-Thomas estimate and obtain ILSE near E_0.

Take $K \in \mathbb{N}$ and decompose the cube $\Lambda_L(x)$ in $\left(\frac{L}{K}\right)^d$ smaller cubes of sidelength K, denoted by $\Lambda_K(j)$, where $j \in \mathcal{J}_L$ for some index set $\mathcal{J}_L$. The probability of finding at least one point $\omega_\gamma \in D_2$ in each cube $\Lambda_{K-2\delta_u}(j)$ such that $\omega_\gamma = 1$ is given by

$$\mathbb{P}\left(\forall j \in \mathcal{J}, \exists \gamma \in \Lambda_{K-\delta_u}(j) \cap D_2 : \omega_\gamma = 1\right) \geq 1 - \left(\frac{L}{K}\right)^d \beta^{C_{R_2,d}(K-2\delta_u)^d}, \tag{4.5.4}$$

where we used the fact $|\Lambda_{K-2\delta_u}(j) \cap D_2| \geq C_{R_2,d} K^d$ uniformly with respect to the center j. Take one of these points in each cube $\Lambda_{K-2\delta_u}(j) \cap D_2$, denote it by γ_j and define the potential V_L as

$$V_L(x) = \sum_{j \in \mathcal{J}_L} u(x - \gamma_j), \tag{4.5.5}$$

so V has exactly one single-site potential in each cube $\Lambda_K(j)$ whose support is at a safety distance from the boundary of $\Lambda_K(j)$, and can be seen as a Delone potential defined on a $(2\delta_u, K)$-Delone set. We have

$$\mathbb{P}\left(V_\omega \geq V\right) \geq 1 - \left(\frac{L}{K}\right)^d \beta^{C_{R_2,d}(K-2\delta_u)^d}. \tag{4.5.6}$$

4.5.1 The case $d \geq 2$.

As in the previous section, consider $\lambda_L(t) = \inf \sigma(H_{t,L}) := \inf \sigma(H_{0,L} + tV_L)$. With a probability given by (4.5.6) we have $\inf \sigma(H_{\omega,L}) \geq \lambda_L(1)$. In turn Theorem **??** applied to the Delone potential V_L of (largest) parameter K yields (see (4.4.6))

$$\inf \sigma(H_{\omega,L}) \geq E_0 + \kappa(K), \tag{4.5.7}$$

where $\kappa(K) = u_- C_{sfUC,d}(K)$, and the constant $C_{sfUC,d}$ is defined in Appendix A.1. From Theorem **??** we know that $C_{sfUC,d}(K) \approx K^{-K^{4/3}}$.

We have proved that

$$\mathbb{P}\left(\inf \sigma(H_{\omega,L}) \geq E_0 + \kappa(K)\right) \geq 1 - \left(\frac{L}{K}\right)^d \beta^{C_{R_2,d}(K-2\delta_u)^d}. \tag{4.5.8}$$

Taking $K = (\log 2L + 2\delta_u)^{1/d+\epsilon}$, for some $\epsilon > 0$, we see that for L larger than some $\mathcal{L} = \mathcal{L}(d, \delta_u, \beta, R_2, \xi)$ the probability in the r.h.s. is $\geq 1 - L^{-\xi}$ for some $\xi > 0$ and, moreover, there exists a constant $C > 0$ such that $K \leq \tilde{K} = (\log CL)^{\frac{1}{d}+\epsilon}$. This yields a gap above E_0 in the spectrum of $H_{\omega,L}$ of size $\kappa(\tilde{K})$, namely,

$$\kappa(\tilde{K}) = (\log CL)^{-\left(\frac{1}{d}+\epsilon\right)(\log CL)^{\frac{4}{3}\left(\frac{1}{d}+\epsilon\right)}}. \tag{4.5.9}$$

As done in the previous section, we apply the Combes-Thomas estimate in the interval $[E_0, E_0 + \kappa(\tilde{K})/2)$. There exist constants c_1 and c_2 such that, for any $x, y \in \mathbb{R}^d$ with $\|x-y\| > L$,

$$\|\chi_y R_{\omega,L_0}(E)\chi_x\| < 2c_1(\log CL)^{\left(\frac{1}{d}+\epsilon\right)(\log 2L)^{4/3}} \exp\left(-\frac{c_2}{\sqrt{2}}(\log CL)^{-\left(\frac{1}{2d}+\frac{\epsilon}{2}\right)(\log CL)^{\frac{4}{3}\left(\frac{1}{d}+\epsilon\right)}} L\right). \tag{4.5.10}$$

In order to obtain a decay of the resolvent, it is enough to have $\frac{4}{3d} < 1$, that is, $d \geq 2$, and $\epsilon > 0$ small enough such that $\frac{4}{3d} + \frac{4\epsilon}{3} < 1$. With this we can start the MSA from [GK11, BoK05].

4.5.2 The case $d = 1$.

In dimension $d = 1$ we can use a unique continuation principle obtained in [KV02a], where its quantitative form is explicitly described in Appendix A.2. This gives the analog of (4.5.9),

$$\kappa(K) = u_- C_{UCP}(K) = u_- \cdot \frac{\delta_u}{K} e^{-2C_{\delta_u,V_0,E}K}, \tag{4.5.11}$$

where the constant $C_{\delta_u,V_0,E}$ is defined in A.2.

Take $K = \frac{1}{N}\log L$, for some positive constant N to be chosen later. Then, the analog of (4.5.8) is

$$\mathbb{P}\left(\inf \sigma(H_{\omega,L}) \geq E_0 + \kappa(K)\right) \geq 1 - \frac{c_3}{\log L} \cdot L^{1-c_4}, \tag{4.5.12}$$

where

$$c_3 := N\beta^{-2C_{R_2}\delta_u}, \quad c_4 := \frac{1}{N} C_{R_2} \left|\log \beta\right| \tag{4.5.13}$$

Here, $C_{R_2} = \frac{1}{R_2}$, so in order to have $c_4 > 1$, we need the following condition on the disorder parameter β:

$$\beta < e^{-NR_2} \tag{4.5.14}$$

in which case, the r.h.s. in (4.5.12) gives the probability we need to start the MSA from [GK11].

Now, we need to make sure that inside the spectral gap, Combes-Thomas estimate gives the desired decay of the resolvent. With our choice of K, the size of the spectral gap above E_0 for $H_{\omega,L}$ is

$$C_{UCP}(L) = \frac{N\delta_u}{\log L} e^{-\frac{2C_{\delta_u,V_0,E}}{N}\log L} = \frac{N\delta_u}{\log L} L^{-2C_{\delta_u,V_0,E}/N} \tag{4.5.15}$$

Consider the energy interval $[E_0, E_0 + \frac{1}{2}C_{UCP}(L)]$. By Combes-Thomas estimate, there exist constants c_1 and c_2 such that

$$\|\chi_y R_{\omega,L}(E)\chi_x\| \leq \frac{c_1 \log L}{N\delta_u} L^{2C_{\delta_u,V_0,E}} \exp\left(-c_2\sqrt{NS}\frac{L}{\sqrt{\log L}L^{\frac{C_{\delta_u,V_0,E}}{N}}}\right). \tag{4.5.16}$$

To obtain the desired decay, we need

$$1 - \frac{C_{\delta_u,V_0,E}}{N} > 0. \tag{4.5.17}$$

Take $N = 2C_{\delta_u,V_0,E}$. Then (4.3.20) decays as $L \cdot e^{-L^{1/4}}$.

With this choice of N, the assumption (4.5.14) on the disorder parameter β becomes

$$\beta < e^{-2C_{\delta_u,V_0,E}R_2}. \tag{4.5.18}$$

Chapter 5

Integrated Density of States for Delone–Anderson operators

Contents

5.1 Randomly coloured point sets

In this chapter, we prove the existence of the Integrated Density of States (IDS) for the Delone–Anderson operators introduced in Chapters 3 and 4, and its relation with their spectral properties. Our results are obtained in the framework of randomly coloured point sets, using an ergodic theorem obtained in [MR12] and under some geometric assumptions on the underlying Delone set. In the case where the background Hamiltonian is the free Laplacian, we show that the IDS exhibits a Lifshitz tails behavior at the bottom of the spectrum. To illustrate the importance of the geometric properties required by D we conclude with an example of a Delone operator with no IDS. Part of these results are contained in the article "Dynamical localization for Delone–Anderson operators" [GMRM], in preparation.

For the convenience of the reader, we recall the framework of dynamical systems for randomly coloured point sets and state the version of the ergodic theorem from [MR12] that we need in our setting. We first recall the definition of a Delone set.

Definition 5.1.1. A subset D of $\mathbb{R}^d$ is called an *(r,R)-Delone set* if it is

- uniformly discrete: there exists a real $r > 0$ such that for any cube Λ_r, $\sharp(D \cap \Lambda_r) \leq 1$, and
- relatively dense: there exists a real $R \geq r > 0$ such that for any cube Λ_R, $\sharp(D \cap \Lambda_R) \geq 1$,

where $\sharp$ stands for cardinality.

Consider the base space $M = \mathbb{R}^d$ and the Abelian group $T = \mathbb{R}^d$ acting on M through the translation α,

$$\alpha : T \times M \to M,\ \alpha(x, \gamma) = x + \gamma \tag{5.1.1}$$

The action is continuous if and only if the map $\alpha(x, m) = (x + m)$ is continuous with respect to the product topology on $T \times M$ and it is said to be *proper* if the application $\alpha(\cdot, m) : T \to M$ is proper, that is, pre-images of compact sets are compact. We say that a group action α is *transitive* if for every $m, m' \in M$, there exists $x \in T$ such that $m + x = m'$ and it is *free* if for any $x \in T$ and $m \in M$ the property $x + m = m$ implies $x = e$, where e is the neutral element in the group T. With the choice of $M = T = \mathbb{R}^d$, as locally compact, second countable topological spaces, α as defined above is clearly transitive, free and proper, and the Euclidian metric dist generating the topology in $\mathbb{R}^d$ is T-invariant and proper. Therefore, this choice of base space M, topological group T and action α satisfies all the requirements in [MR12, Section 2] (see Assumption 2.1 and Example 2.5 therein).

Note that the Lebesgue measure is inversion invariant with respect to the group T, that is, for any measurable function we have

$$\int_T f(-x)dx = \int_T f(x)dx, \text{ and in particular, } |-S| = |S| \text{ for } S \subset T, \tag{5.1.2}$$

where $-S = \{x \in T : \exists s \in S \text{ s.t. } x + s = 0\}$. In this context, T is called *unimodular*, which is needed to apply [MR12, Theorem 3.11].

We denote the collection of all subsets of $\mathbb{R}^d$ which are uniformly discrete of parameter r by $\mathcal{P}_r(\mathbb{R}^d)$.

Definition 5.1.2. We define the *vague topology* on $\mathcal{P}_r(\mathbb{R}^d)$ as the weakest topology such that the function $f_\varphi : \mathcal{P}_r(\mathbb{R}^d) \to \mathbb{R}$ defined by $f_\varphi(P) = \sum_{p \in P} \varphi(P)$ is continuous for every φ in $\mathcal{C}_c(\mathbb{R}^d)$. This topology can be seen as generated by the metric given by

$$d(P, P') := \min\left\{\frac{1}{2}, \inf_{\epsilon > 0}\left\{P \cap B_{\frac{1}{\epsilon}}(m) \subset (P')_\epsilon \text{ and } P' \cap B_{\frac{1}{\epsilon}}(m) \subset (P)_\epsilon\right\}\right\}, \tag{5.1.3}$$

for some fixed $m \in \mathbb{R}^d$, where $B_a(x)$ is the ball of radius a around $x \in \mathbb{R}^d$. In the r.h.s. of (5.1.3) we use the notation $(A)_\epsilon := \{m' \in \mathbb{R}^d : \operatorname{dist}(m', A) \leq \epsilon\}$ for the *thickened* version of a set $A \subset \mathbb{R}^d$.

The set $\mathcal{P}_r(\mathbb{R}^d)$ is a compact space with respect to the vague topology (see [MR12, Proposition 2.6])

Definition 5.1.3. For $D \subset \mathbb{R}^d$ a Delone set, we define its closed $\mathbb{R}^d$–orbit as

$$X_D := \overline{\{x + D : x \in \mathbb{R}^d\}} \subset \mathcal{P}_r(\mathbb{R}^d), \tag{5.1.4}$$

where $x + D := \{x + \gamma : \gamma \in D\}$ and the closure is taken with respect to the vague topology. In particular, the orbit X_D is compact with respect to this topology. We denote an arbitrary element of X_D by P.

The induced translation action α_{X_D} on X_D,

$$\alpha_{X_D} : T \times X_D \to X_D \tag{5.1.5}$$
$$\alpha(x, P) = x + P \quad \text{for } P \in X_D, \tag{5.1.6}$$

is continuous. The triple consisting of X_D, the group $\mathbb{R}^d$ and the action α_{X_D} constitutes a compact dynamical system.

Now, consider a Borel-measurable subset $\mathbb{A} \subset \mathbb{R}$ as *colour space* . For a relatively discrete point set $P \in \mathcal{P}_r(\mathbb{R}^d)$, we introduce the product space

$$\Omega_P := \bigotimes_{p \in P} \mathbb{A}, \tag{5.1.7}$$

of its possible colour realizations $\omega = (\omega_p)_{p \in P}$, where $\omega_p \in \mathbb{A}$ for all $p \in P$. Define the randomly coloured point set P^ω with colour realization $\omega \in \Omega_P$ as

$$P^\omega := \{(p, \omega_p) : p \in P, \omega \in \Omega_P\}, \tag{5.1.8}$$

belonging to the coloured orbit

$$\hat{X}_D := \{P^\omega : P \in X_D, \omega \in \Omega_P\} = \overline{\{x + D^\omega : x \in \mathbb{R}^d\}}, \tag{5.1.9}$$

of a Delone set $D \in \mathcal{P}_r(\mathbb{R}^d)$. The sets equality in the r.h.s. is proved in [MR12, Lemma 3.6]. The product space $\hat{\mathbb{R}}^d := \mathbb{R}^d \times \mathbb{A}$ equipped with the product topology is a base space for $\hat{X}_D$. The induced continuous translation $\hat{\alpha} : T \times \hat{\mathbb{R}}^d \to \hat{\mathbb{R}}^d$ acts on $\hat{X}_D$ as $x + P^\omega = (x + P)^{\tau_x(\omega)}$ where $\tau_x(\omega)$ is defined by

$$\tau_x : \begin{array}{ccc} \Omega_P & \to & \Omega_{x+P} \\ (\omega_p)_{p \in P} & \mapsto & (\omega_p)_{x+p \in x+P} \end{array}. \tag{5.1.10}$$

This means that the colour ω_p is translated along with p. Furthermore, the closure in the r.h.s. of (5.1.9) is taken with respect to the vague topology on the space of relatively discrete coloured point sets $\mathcal{C}_r(\mathbb{R}^d) := \{P^\omega : P \in \mathcal{P}_r(\mathbb{R}^d), \omega \in \Omega_P\}$. This is the coarsest topology such that for every function $\varphi \in C_c(\hat{\mathbb{R}}^d)$ the map $\mathcal{C}_r(\mathbb{R}^d) \ni P^\omega \mapsto \sum_{p \in P} \varphi(p, \omega_p)$ is continuous.

The above defined translation of a coloured point set is a continuous map on $\mathcal{C}_r(\mathbb{R}^d)$ [MR12, Lemma 3.6], and the coloured orbit $\hat{X}_D$ is compact in the vague topology for every (r, R)-Delone set $D \subset \mathbb{R}^d$ [MR12, Prop. 3.5]. We note that the triple consisting of $\hat{X}_D$, the group $\mathbb{R}^d$ and its continuous translation action on $\hat{X}_D$ is a compact topological dynamical system. Given any point set P, let $\mathcal{A}$ be the Borel σ-algebra on $\mathbb{A}$ and let $\mathbb{P}$ be the Borel probability measure on $(\mathbb{A}, \mathcal{A})$. We consider the product measure

$$\mathbb{P}_P := \bigotimes_{p \in P} \mathbb{P} \tag{5.1.11}$$

acting on $(\Omega_P, \mathcal{A}_P)$, where $\mathcal{A}_P$ is the product Borel σ-algebra on Ω_P. It follows from [MR12, Lemma 3.9 (i)] that the family $(\mathbb{P}_P)_{P \in X_D}$ satisfies the properties:

i. $\mathbb{R}^d$- *covariance*: $\mathbb{P}_{x+P} = \mathbb{P}_P \otimes \tau_x^{-1}$ for all $x \in \mathbb{R}^d$ and all $P \in X_D$.

ii. *Independence at a distance*: There exists a length $\rho > 0$ such that for every $P \in X_D$ and every $B_1, B_2 \subset M$ with $\text{dist}(B_1, B_2) > \rho$ the local σ-algebras $\mathcal{A}_P|_{B_1}, \mathcal{A}_P|_{B_2}$ are independent.

iii. *Compatibility*: For every continuous function f on $\hat{X}_P$ the colour average $E_f : X_D \to \mathbb{R}$,

$$E_f(P) := \int_{\Omega_P} f(P^\omega) d\mathbb{P}_P(\omega) \tag{5.1.12}$$

is a measurable continuous function on X_D.

An increasing sequence of non-empty compact subsets $(\Lambda_L)_{L\in\mathbb{N}}$ of $\mathbb{R}^d$ such that $\cup_L \Lambda_L = \mathbb{R}^d$ is called a *Følner* sequence if, for every compact $K \subset \mathbb{R}^d$ we have

$$\lim_{L\to\infty} \frac{|\delta^K \Lambda_L|}{|\Lambda_L|} = 0 \tag{5.1.13}$$

where δ^K is the symmetric difference of Λ_L,

$$\delta^K := (K + \Lambda) \setminus \Lambda \cup (K + \Lambda)^c \setminus \Lambda^c, \quad \text{ with } A^c = \mathbb{R}^d \setminus A, \tag{5.1.14}$$

and $K + \Lambda_L = \{k + x : k \in K \text{ and } x \in \Lambda_L\}$. Moreover, it is called *tempered* if there exists a constant $C \geq 1$ such that for all $L \in \mathbb{N}$ we have

$$\left| \bigcap_{k=1}^{L-1} \Lambda_L - \Lambda_k \right| \leq C \, |\Lambda_L| \, . \tag{5.1.15}$$

In the following we consider a sequence of concentric cubes $\Lambda_{z,L}$ of sidelength L centered in $z \in \mathbb{R}^d$, that exhausts $\mathbb{R}^d$ when L tends to infinity. This sequence, used to define the finite-volume versions of Hamiltonians in the previous chapters, is a tempered Følner sequence. For $z = 0$, we simply write Λ_L.

Remark 5.1.4. In the literature of Delone dynamical systems, definitions are often expressed in terms of *van Hove* sequences, which is a slightly more restrictive notion. With our choice of the base space and the topological group acting on it, the sequence of cubes $(\Lambda_L)_{L\in\mathbb{N}}$ is both a tempered Følner and a tempered van Hove sequence.

Before we state the version of the ergodic theorem [MR12, Thm. 3.11] that we need in our setting, we remark that on every compact topological dynamical system there exists an ergodic Borel probability measure, cf. [Wal82, Section 6.2]. If this measure is unique, then the dynamical system is called *uniquely ergodic*.

Theorem 5.1.5. *Let $D \subset \mathbb{R}^d$ be a Delone set and let μ be an ergodic Borel probability measure on X_D. Then there exists an ergodic probability measure $\hat{\mu}$ on $\hat{X}_D$, which is uniquely determined by μ, such that the following holds.*

(i) For every $f \in L^1(\hat{X}_D, \hat{\mu})$ we have

$$\int_{\hat{X}_D} f(P^\omega) \, d\hat{\mu}(P^\omega) = \int_{X_D} \left(\int_{\Omega_P} f(P^\omega) \, d\mathbb{P}_P(\omega) \right) d\mu(P). \tag{5.1.16}$$

(ii) For every $f \in L^1(\hat{X}_D, \hat{\mu})$ the limit

$$\lim_{L\to\infty} \frac{1}{L^d} \int_{\Lambda_L} f(x + \tilde{P}^{\tilde{\omega}}) \, dx = \int_{\hat{X}_D} f(P^\omega) \, d\hat{\mu}(P^\omega) \tag{5.1.17}$$

exists for $\hat{\mu}$-a.e. $\tilde{P}^{\tilde{\omega}} \in \hat{X}_D$. Moreover, if X_D is even uniquely ergodic and if f is continuous, then the limit (5.1.17) *exists* for every *$\tilde{P} \in X_D$ and for $\mathbb{P}_{\tilde{P}}$-a.a. $\tilde{\omega} \in \Omega_{\tilde{P}}$.*

The ergodic theorem will be most useful in the uniquely ergodic situation and for continuous f. In this case the limit in (5.1.17) exists for $\tilde{P} = D$, the Delone set we started with, and for $\mathbb{P}_D$-a.e. $\tilde{\omega} \in \Omega_D$.

The relation between the geometric properties of D and the ergodic properties of X_D has been widely studied in the literature (see [MR12, Section 2.3], [LP03]). To understand the conditions a Delone set has to satisfy in order to obtain the optimal result from Theorem 5.1.5, we recall some relevant notions from the theory of Delone dynamical systems (DDS):

Definition 5.1.6. i) Given a Delone set $D \in \mathcal{P}_r(\mathbb{R}^d)$, any finite subset $Q \subset D$ is called a *pattern* of D, the compact subset $A \subset \mathbb{R}^d$ such that $Q = A \cap D$ is called the support of Q, and Q is also called an A-pattern. Two sets $A, A' \subset \mathbb{R}^d$ (resp. two patterns Q, Q') are called *equivalent* if there exists $x \in \mathbb{R}^d$ such that $x + A = A'$ (resp. $x + Q = Q'$).

ii) We say D is of *finite local complexity* if for every compact set $A \subset \mathbb{R}^d$, there exists a finite collection of patterns, denoted by $\mathcal{F}$, such that every pattern of D with support equivalent to A, is equivalent to some pattern in $\mathcal{F}$. That is, for any given compact set $A \subset \mathbb{R}^d$, there are finitely many A-patterns in D.

iii) A Delone set D of finite local complexity is called *linearly repetitive* if there exists a constant C such that for all $R > 0$, the ball of radius CR in D, $B_{CR} \cap D$, contains every possible B_R-pattern.

iv) Let $(\Lambda_L)_{L \in \mathbb{N}}$ be a sequence of concentric cubes of sidelength L in $\mathbb{R}^d$. We define the *pattern frequency* of Q in D as the following limit, if it exists,

$$\eta(Q) := \lim_{L \to \infty} \frac{\sharp\{\tilde{Q} \subset D \,:\, \exists x \in (-\Lambda_L) \text{ s.t. } x + Q = \tilde{Q}\}}{|\Lambda_L|}, \tag{5.1.18}$$

that is, the number of equivalent patterns of Q in D per volume converges (it is known that is quantity is always bounded, so the question is to know whether the equality $\liminf = \limsup$ holds [MR12, Lemma 2.25]).

v) We say that D has *uniform pattern frequency* if for any pattern $Q \subset D$ the sequence

$$\frac{\eta_{x,L}(Q)}{|\Lambda_L|} := \frac{\sharp\{\tilde{Q} \subset D \,:\, \exists y \in (x + \Lambda_L) \text{ s.t. } y + \tilde{Q} = Q\}}{|\Lambda_L|} \tag{5.1.19}$$

converges uniformly with respect to $x \in T$, when L goes to infinity. Moreover, we say that D has a *strict* uniform pattern frequency if this limit is strictly positive.

If D is of finite local complexity and has uniform pattern frequencies, then X_D is uniquely ergodic [MR12, Proposition 2.32], so we have the following corollary

Corollary 5.1.7. *Let D be a Delone set of finite local complexity and with the property of uniform patter frequencies. If the function f in Theorem 5.1.5 is continuous, the limit in* (5.1.17) *exists for every $P \in X_D$ and for a.e. $\omega \in \Omega_P$.*

Remark 5.1.8. In particular, with our choice of $M = T = \mathbb{R}^d$ and α, if D is *linearly repetitive*, then it has *strict* uniform pattern frequencies [LP03, Theorem 6.1], and the previous Corollary holds .

5.2 Existence of the IDS for the Delone–Anderson model

Given a coloured Delone set $\mathcal{C}_r(\mathbb{R}^d)$, consider the Hamiltonian

$$H_{D^\omega} = H_0 + V_{D^\omega} \quad \text{on } \mathrm{L}^2(\mathbb{R}^d), \tag{5.2.1}$$

The background operator is either $H_0 = -\Delta$ or, if $d = 2$, we also allow for the Landau Hamiltonian with constant magnetic field $B > 0$ defined by

$$H_0 = (-i\nabla - \mathbf{A})^2 \quad \text{with } \mathbf{A} = \frac{B}{2}(x_2, -x_1). \tag{5.2.2}$$

The random potential V_{D^ω} is defined by

$$V_{D^\omega} = \sum_{\gamma \in D} \omega_\gamma u(x - \gamma), \tag{5.2.3}$$

where $(\omega_\gamma)_{\gamma \in D}$ are independent identically distributed random variables of common probability distribution ρ with $\operatorname{supp} \rho = [0, M]$, for some fixed $M > 0$. The single-site potential satisfies property (v1) (4.1.2). Then, H_{D^ω} can be seen as a function of a randomly coloured Delone set $D^\omega \in \hat{X}_D$ with colour space $\mathbb{A} = [0, M]$, in the context of Section 5.1.

We write $P_E(H_{D^\omega}) := \chi_{(-\infty,E]}(H_{D^\omega})$ for the spectral projection associated to H_{D^ω} in the interval $(-\infty, E] \subset \mathbb{R}$, where $E \in \mathbb{R}$. Since H_{D^ω} is lower semi–bounded, $P_E(H_{D^\omega}) = \chi_{[E_0,E]}(H_{D^\omega})$ where $E_0 = \inf \sigma(H_{D^\omega}) \geq 0$. More generally, consider a continuous and compactly supported function $F \in C_c(\mathbb{R})$. For our choices of H_0 and the scalar random potential V_{D^ω}, the operator $F(H_{D^\omega})\, \chi_{\Lambda_{z,L}}$ is trace class for every $z \in \mathbb{R}^d$ and $L > 0$, where $\chi_{\Lambda_{z,L}}$ stands for the characteristic function of the cube $\Lambda_{z,L}$. Moreover, it follows from [BrLM04, Thm. 1.14(i)] that this operator has a bounded continuous integral kernel $f(H_{D^\omega}) \in C(\mathbb{R}^d \times \mathbb{R}^d) \cap \mathrm{L}^\infty(\mathbb{R}^d \times \mathbb{R}^d)$ for almost every $\omega \in \Omega$. Its trace is given by

$$\operatorname{tr}\, \big(F(H_{D^\omega})\, \chi_{\Lambda_{z,L}}\big) = \int_{\Lambda_{z,L}} f(H_{D^\omega})(x, x)\, dx, \tag{5.2.4}$$

which follows e.g. from [BrLM04, Cor. 1.16 and 1.18].

We say that a Delone set $D \subset \mathbb{R}^d$ is uniquely ergodic, if the associated dynamical system X_D has this property.

Theorem 5.2.1. *Fix $F \in C_c(\mathbb{R})$ and consider a uniquely ergodic Delone set $D \subset \mathbb{R}^d$. Then there exists a measurable subset $\tilde{\Omega}_D \subseteq \Omega_D$ (depending on F) of full probability, $\mathbb{P}_D(\tilde{\Omega}_D) = 1$, such that for every $\tilde{\omega} \in \tilde{\Omega}_D$ and every $z \in \mathbb{R}^d$ the limit*

$$\begin{aligned}\lim_{L\to\infty} \frac{1}{L^d} \operatorname{tr}\, \big(F(H_{D^{\tilde{\omega}}})\chi_{\Lambda_{z,L}}\big) &= \int_{\hat{X}_D} f(H_{P^\omega})(0,0)\, d\hat{\mu}(P^\omega) \\ &= \int_{X_D} \Big(\int_{\Omega_P} f(H_{P^\omega})(0,0)\, d\mathbb{P}_P(\omega)\Big)\, d\mu(P)\end{aligned} \tag{5.2.5}$$

exists and is independent of $\tilde{\omega} \in \tilde{\Omega}_D$ and $z \in \mathbb{R}^d$. Here, μ is the unique ergodic measure on the compact uncoloured orbit X_D and $\hat{\mu}$ is given by Theorem 5.1.5.

Proof. Let $\{U_a\}$ be the family of unitary operators on $\mathrm{L}^2(\mathbb{R}^d)$ associated to translations by $a \in \mathbb{R}^d$, that is, $U_a\psi := \psi(\cdot - a)$ for every $\psi \in L^2(\mathbb{R}^d)$. (In the case $\mathbf{A} \neq 0$ we use magnetic translations.) Recalling the shifts on the probability space (5.1.10), we find

$$U_a H_{D^\omega} U_a^* = H_{(a+D)^{\tau_a(\omega)}} = H_{a+D^\omega}, \tag{5.2.6}$$

and thus $F(H_{a+D^\omega}) = U_a F(H_{D^\omega}) U_a^*$. In turn, this implies

$$f(H_{a+D^\omega}) = f(H_{D^\omega})(\cdot - a, \cdot - a) \tag{5.2.7}$$

for the integral kernels, and in particular $f(H_{D^\omega})(x,x) = f(H_{-x+D^\omega})(0,0)$ for every $x \in \mathbb{R}^d$. Now we define the map $\hat{X}_D \ni P^\omega \mapsto \Phi(P^\omega) := f(H_{P^\omega})(0,0)$. We conclude from (5.2.4) that

$$\operatorname{tr}\left(F(H_{D^\omega})\,\chi_{\Lambda_{z,L}}\right) = \int_{\Lambda_{-z,L}} \Phi(x + D^\omega)\,dx. \tag{5.2.8}$$

By Lemma 5.2.2, below, the map Φ is continuous. The claim for fixed $z \in \mathbb{R}^d$ now follows from Theorem 5.1.5.(ii). To complete the proof we note that the choice of $z \in \mathbb{R}^d$ does not affect the convergence because the function $\mathbb{R}^d \ni x \mapsto \Phi(x + D^\omega)$ is bounded and because for every $z, z' \in \mathbb{R}^d$ the Lebesgue volume of the symmetric difference $\Lambda_{z,L} \triangle \Lambda_{z',L}$ behaves like $\mathcal{O}(L^{d-1})$ as $L \to \infty$. □

Lemma 5.2.2. *Let $f(H_{P^\omega})(x,y)$ be the continuous bounded integral kernel associated to the operator $F(H_{P^\omega})$, for $F \in \mathcal{C}_c(\mathbb{R}^d)$. The map $\Phi(P^\omega) : \hat{X}_D \to \mathbb{R}$ defined by $\Phi(P^\omega) := f(H_{P^\omega})(0,0)$, is continuous on $\hat{X}_D$.*

Proof. As F satisfies the condition [BrLM04, Eq. 1.20] for some constants $\gamma, \tau \in]0,\infty[$ depending on $\|F\|_\infty$ and $\operatorname{supp} F$, we have that the kernel $f(H_{P^\omega})(0,0)$, for a fixed P^ω is given by the formula

$$f(H_{P^\omega})(0,0) = \langle k_t^{H_{P^\omega}}(\cdot,0), e^{2tH_{P^\omega}} F(H_{P^\omega}) k_t^{H_{P^\omega}}(\cdot,0)\rangle, \tag{5.2.9}$$

for an arbitrary $t \in]0, \tau/2[$, where

$$k_t^{H_{P^\omega}}(\cdot,0) := \frac{e^{-|x|^2/2t}}{(2\pi t)^{d/2}} \int \mu_{x,0}^{0,t}(db) e^{-S_t(A,V_{P^\omega};b)}. \tag{5.2.10}$$

Here S_t is the Euclidian action functional defined in Appendix A.4, together with a recall of the framework of [BrLM04].

Take an arbitrary element $Q^{\tilde{\omega}} \in \hat{X}_D$, and a sequence $\{{P^\omega}_n := (P_n)^{\omega_n}\}_{n\in\mathbb{N}} \subset \hat{X}_D$ that converges to $Q^{\tilde{\omega}}$ in the product topology of $\hat{X}_D$. Write

$$H_n = H_0 + V_{P^\omega_n}, \quad H = H_0 + V_{Q^{\tilde{\omega}}}. \tag{5.2.11}$$

Now, since $V_{{P^\omega}_n}$ satisfies the required properties in [BrLM04], uniformly on n and $k_t^H(\cdot,0) \in \mathrm{L}^\infty_G(\mathbb{R}^d) \subset \mathrm{L}^2_G(\mathbb{R}^d)$ (see [BrLM04, Eq. 1.7]), then (see [BrLM04, Theorem 1.10]),

$$k_t^H(\cdot,0) \in \mathcal{D}(e^{2tH_n}) \quad \forall n, \tag{5.2.12}$$

we then have, using the Cauchy-Schwartz inequality,

$$\begin{aligned} |f(H_n)(0,0) - f(H)(0,0)| &\leq \left|\langle k_t^{H_n}(\cdot,0), e^{2tH_n} F(H_n) \left(k_t^{H_n}(\cdot,0) - k_t^{H}(\cdot,0)\right)\rangle\right| \\ &\quad + \left|\langle k_t^{H_n}(\cdot,0), \left(e^{2tH_n} F(H_n) - e^{2tH} F(H)\right) k_t^{H_n}(\cdot,0)\rangle\right| \\ &\leq \|k_t^{H_n}(\cdot,0)\|_2 \, \|e^{2tH_n} F(H_n)\| \, \|k_t^{H_n}(\cdot,0) - k_t^{H}(\cdot,0)\|_2 \\ &\quad + \|k_t^{H_n}(\cdot,0)\|_2 \, \| \left(e^{2tH_n} F(H_n) - e^{2tH} F(H)\right) k_t^{H_n}(\cdot,0)\|_2 \end{aligned} \tag{5.2.13}$$

In order to show the r.h.s. tends to 0 as n tends to infinity, note that

i. From (5.2.10) we see that if $v_-(P^\omega) := \inf_{x\in\mathbb{R}^d} V_{P^\omega}(x)$, then we have the bound

$$\left|k_t^{H_n}(x,0)\right| \leq \frac{e^{-|x|^2/2t}}{(2\pi t)^{d/2}} e^{t|v_-(P^\omega)|}. \tag{5.2.14}$$

In our case, $v_-({P^\omega}_n) = 0$ uniformly in n, and therefore $\|k_t^{H_n}(\cdot,0)\|_2 < \infty$, for all $n \in \mathbb{N}$.

ii. The operator $e^{2tH_n} F(H_n)$ is bounded for all $n \in \mathbb{N}$, since F has compact support.

iii. The fact that $\| \left(e^{2tH_n} F(H_n) - e^{2tH} F(H)\right) k_t^{H_n}(\cdot,0)\|_2 \to 0$ as n goes to infinity comes from the convergence of these operators in the strong resolvent sense.

It remains to show that $e^{2tH_n} F(H_n) \to e^{2tH} F(H)$ in the strong resolvent sense and that $\|k_t^{H_n}(\cdot,0) - k_t^{H}(\cdot,0)\|_2$ tends to zero as $n \to \infty$.

Strong Resolvent Convergence of $e^{2tH_n} F(H_n)$.

Since $e^{2tx} F(x)$ is a bounded and continuous function on $\mathbb{R}$, by [RSI, Theorem VIII.20] it is enough to prove that H_n converges to H in the strong resolvent sense, that is,

$$\lim_{n\to\infty} \| \left(R_{H_n}(z) - R_H(z)\right) \varphi\| = 0, \quad \forall \varphi \in \mathrm{L}^2(\mathbb{R}^d),\ \mathrm{Im} z \neq 0, \tag{5.2.15}$$

By [RSI, Theorem VIII.19], it is sufficient to prove this for a fixed $z \in \mathbb{C}$ with $\mathrm{Im} z \neq 0$ and for $\varphi \in \mathcal{C}_c^\infty(\mathbb{R}^d)$, since $\mathcal{C}_c^\infty(\mathbb{R}^d)$ is dense in $\mathrm{L}^2(\mathbb{R}^d)$, and noticing that all operators $R_{H_n}(z)$ are bounded by $1/|\mathrm{Im} z|$.

Consider a function $\varphi \in \mathcal{C}_c^\infty(\mathbb{R}^d)$, then there exists $n_0 \in \mathbb{N}$ such that $\mathrm{supp}\,\varphi \subset B_{n_0}$ and $\chi_{B_{n_0}} \varphi = \varphi$, where $\chi_{B_{n_0}}$ is the characteristic function of the ball $B_{n_0} \subset \mathbb{R}^d$ (since these results are uniform with respect to the center of the ball, we omit it from the notation). Take the sequence ${P^\omega}_n$ approximating $Q^{\tilde\omega}$, such that

$$\mathrm{dist}_{\hat{X}_D}({P^\omega}_n, Q^{\tilde\omega}) < \frac{1}{n}, \quad \forall n \geq n_0. \tag{5.2.16}$$

We then have, taking $m = n + n_0$, for some $n \in \mathbb{N}$,

$$\| \left(R_{H_m}(z) - R_H(z)\right) \varphi\| \leq \frac{1}{|\mathrm{Im}z|}\| \left(V_{Q^{\tilde{\omega}}} - V_{P^{\omega}{}_m}\right) R_H(z)\varphi\| \tag{5.2.17}$$

$$= \frac{1}{|\mathrm{Im}z|}\| \left(V_{Q^{\tilde{\omega}}} - V_{P^{\omega}{}_m}\right) \left(\chi_{B_m} + \chi_{B_m^c}\right) R_H(z) B_{n_0}\varphi\| \tag{5.2.18}$$

$$\leq \frac{1}{|\mathrm{Im}z|^2}\| \left(V_{Q^{\tilde{\omega}}} - V_{P^{\omega}{}_m}\right) \chi_{B_m}\|_\infty + \tag{5.2.19}$$

$$\frac{1}{|\mathrm{Im}z|}\| \left(V_{Q^{\tilde{\omega}}} - V_{P^{\omega}{}_m}\right) \chi_{B_m^c} R_H(z) B_{n_0}\varphi\|. \tag{5.2.20}$$

To estimate the first term in (5.2.20), notice that (5.2.16) with $\mathrm{dist}_{\hat{X}_D} = \max\{\mathrm{dist}_{X_D}, \mathrm{dist}_{\mathbb{A}}\}$ implies

$$\mathrm{dist}_{X_D}(P^{\omega}{}_m, Q^{\tilde{\omega}}) < \frac{1}{m} \quad \text{and} \quad \mathrm{dist}_{\mathbb{A}}(\omega_m, \tilde{\omega}) < \frac{1}{m}, \tag{5.2.21}$$

where $\mathbb{A} = [0, M]$ is the colour space, and $\max\{\|\omega_m\|_\infty, \|\omega\|_\infty\} \leq M$. Therefore,

$$\| \left(V_{Q^{\tilde{\omega}}} - V_{P^{\omega}{}_m}\right) \chi_{B_m}\|_\infty = \| \sum_{\gamma \in Q \cap B_m} \tilde{\omega}_\gamma u(x-\gamma) - \sum_{j \in P_m \cap B_m} \omega_{m,j} u(x-j)\|_\infty \tag{5.2.22}$$

$$\leq \| \sum_{\gamma \in Q \cap B_m} (\tilde{\omega}_\gamma - \omega_{m,j}) u(x-\gamma)\|_\infty + \tag{5.2.23}$$

$$\| \sum_{\gamma \in P \cap B_m} \omega_{m,j}(u(x-\gamma) - u(x-j))\|_\infty \tag{5.2.24}$$

$$\leq \frac{1}{m} + M \sup_{\substack{j \in (Q)_{1/m} \cap B_m \\ \gamma \in (P_m)_{1/m} \cap B_m}} |u(x-\gamma) - u(x-j)|, \tag{5.2.25}$$

where we used the definition of dist_{X_D} and that $\| \sum_\gamma u(\cdot - \gamma)\|_\infty \leq 1$. If we assume that u is continuous in some neighborhood of the origin, the second term in the r.h.s. of (5.2.32) will decay as C/m.

As for the second term in the r.h.s of (5.2.20), the Combes-Thomas estimate yields

$$\frac{1}{|\mathrm{Im}z|}\| \left(V_{Q^{\tilde{\omega}}} - V_{P^{\omega}{}_m}\right) \chi_{B_m^c} R_H(z) B_{n_0}\varphi\| \leq \frac{2M}{|\mathrm{Im}z|\ |z|} e^{-c_2\sqrt{|\mathrm{Im}z|}\, n}. \tag{5.2.26}$$

Replacing (5.2.32) and (5.2.26) in (5.2.20) and taking n (and therefore m)$\to \infty$, and recalling that this can be done for every n_0, we obtain

$$\lim_{n \to \infty} \| \left(R_{H_n}(z) - R_H(z)\right) \varphi\| = 0, \quad \forall \varphi \in \mathcal{C}_c^\infty(\mathbb{R}^d),\ \mathrm{Im}z \neq 0. \tag{5.2.27}$$

L^2-convergence of $k_t^{H_n}(\cdot, 0)$.

$$\|k_t^{H_n}(\cdot, 0) - k_t^H(\cdot, 0)\|_2 = \int_{\mathbb{R}^d} \left|k_t^{H_n}(x, 0) - k_t^H(x, 0)\right|^2 dx \tag{5.2.28}$$

Using the expression (5.2.10) we have

$$\begin{aligned}\left|k_t^{H_n}(x,0)-k_t^H(x,0)\right| &\le \frac{e^{-|x|^2/(2t)}}{(2\pi t)^{d/2}}\int \mu_{x,0}^{0,t}(db)\left|e^{-S_t(A,V_{P_n^\omega};b)}-e^{-S_t(A,V_{Q^{\tilde\omega}};b)}\right| \\ &\le \frac{e^{-|x|^2/(2t)}}{(2\pi t)^{d/2}}\int \mu_{x,0}^{0,t}(db)\left|e^{-S_t(0,V_{P_n^\omega};b)}-e^{-S_t(0,V_{Q^{\tilde\omega}};b)}\right|. \qquad (5.2.29)\end{aligned}$$

Now we use the inequality $|e^z-e^{z'}|\le|z-z'|\,e^{\max\{z,z'\}}$ for all $z,z'\in\mathbb{R}$ plus the fact that $\max\{\|V_{P_n^\omega}\|_\infty,\|V_{Q^{\tilde\omega}}\|_\infty\}\le M$, and obtain,

$$\begin{aligned}\int \mu_{x,0}^{0,t}(db)\left|e^{-S_t(0,V_{P_n^\omega};b)}-e^{-S_t(0,V_{Q^{\tilde\omega}};b)}\right| &\le \int \mu_{x,0}^{0,t}(db)\left|S_t(0,V_{P_n^\omega};b)-S_t(0,V_{Q^{\tilde\omega}};b)e^{Mt}\right| \\ &= e^{Mt}\int \mu_{x,0}^{0,t}(db)\left|S_t(0,V_{P_n^\omega}-V_{Q^{\tilde\omega}};b)\right| \\ &\le e^{Mt}\int \mu_{x,0}^{0,t}(db)\left|S_t(0,\left(V_{P_n^\omega}-V_{Q^{\tilde\omega}}\right)\Theta(n-|x|);b)\right| \\ &+e^{Mt}\int \mu_{x,0}^{0,t}(db)\left|S_t(0,\left(V_{P_n^\omega}-V_{Q^{\tilde\omega}}\right)\Theta(|x|-n);b)\right|, \\ &\qquad (5.2.30)\end{aligned}$$

where we used the Heaviside function notation from [BrLM04]. Note that for n big enough we have (see (5.2.22))

$$\|\left(V_{P_n^\omega}-V_{Q^{\tilde\omega}}\right)\Theta(n-|x|)\|_\infty\le\frac{1}{n}. \qquad (5.2.31)$$

Then the first term in the r.h.s. of (5.2.30) is bounded by

$$\int \mu_{x,0}^{0,t}(db)\left|S_t(0,\left(V_{P_n^\omega}-V_{Q^{\tilde\omega}}\right)\Theta(n-|x|);b)\right|\le\frac{t}{n}. \qquad (5.2.32)$$

As for the second term in the r.h.s. of (5.2.30), the proof is similar to that of [BrLM04, Lemma 2.2]. One uses $\Theta(|x|-n)=\Theta(\left|\frac{x}{n}\right|-1)\le\frac{|x|^2}{n^2}$ together with the boundedness of both $V_{P_n^\omega}$ and $V_{Q^{\tilde\omega}}$ to conclude that

$$\int \mu_{x,0}^{0,t}(db)\left|S_t(0,\left(V_{P_n^\omega}-V_{Q^{\tilde\omega}}\right)\Theta(|x|-n);b)\right|\le\frac{2M}{n^2}\int \mu_{x,0}^{0,t}(db)\int_0^t ds\,|b(s)|^2. \qquad (5.2.33)$$

Standardizing the Brownian bridge according to $b(s)=t^{1/2}\tilde b(s/t)+x-xs/t$, we see that Fubini's theorem yields

$$\begin{aligned}\frac{2M}{n^2}\int \mu_{x,0}^{0,t}(d\tilde b)\int_0^t ds\,|b(s)|^2 &\le \frac{2M}{n^2}\int_0^1 du\int \mu_{0,0}^{0,1}(d\tilde b)|t^{1/2}\tilde b(u)+x-xu|^2 \qquad (5.2.34)\\ &\le \frac{4Mt}{n^2}\int_0^1 du\int \mu_{0,0}^{0,1}(d\tilde b)\left(t|\tilde b(u)|^2+|x|^2(1+u^2)\right) \qquad (5.2.35)\\ &\le \frac{4Mt}{n^2}\left(t\int_0^1 du\int \mu_{0,0}^{0,1}(d\tilde b)|\tilde b(u)|^2\right)+\frac{16Mt}{3n^2}|x|^2. \qquad (5.2.36)\\ &\qquad (5.2.37)\end{aligned}$$

In the last inequality, by writing $|\tilde{b}(u)|^2 = \sum_{1\leq j\leq d} |\tilde{b}_j(u)|^2$, the first term can be bounded by dt using the variance of the components $\tilde{b}_j(u)$, $0 \leq j \leq d$, which is bounded by t for all j (see Appendix A.4).

Recalling $t \in]0, \tau/2[$, we have the following bound,

$$\int \mu_{x,0}^{0,t}(db) \left| S_t(0, \left(V_{P_n^\omega} - V_{Q^{\tilde{\omega}}}\right) \Theta(|x| - n); b)\right| \leq \frac{C(M,\tau)}{n^2}(1 + |x|^2), \tag{5.2.38}$$

for some constant $C(M,\tau)$ depending on M and τ. Plugging this together with (5.2.32) into (5.2.30) gives, for (5.2.29),

$$\left|k_t^{H_n}(x,0) - k_t^H(x,0)\right| \leq \frac{e^{-|x|^2/(2t)}}{(2\pi t)^{d/2}} \frac{C'(M,\tau)}{n}(1 + |x|^2),\ \forall t \in]0, \tau/2[\tag{5.2.39}$$

$$\left|k_t^{H_n}(x,0) - k_t^H(x,0)\right|^2 \leq \frac{e^{-|x|^2/t}}{(2\pi t)^{d}} \frac{C'(M,\tau)^2}{n^2}(1 + |x|^2)^2, \tag{5.2.40}$$

for some constant $C'(M,\tau)$ depending on M and τ. Since the r.h.s. is in $\mathrm{L}^2(\mathbb{R}^d)$, we have that for fixed t and n,

$$\|k_t^{H_n}(\cdot,0) - k_t^H(\cdot,0)\|_2^2 = \int_{\mathbb{R}^d} \left|k_t^{H_n}(x,0) - k_t^H(x,0)\right|^2 dx \leq \frac{C''(M,\tau,t)^2}{n^2}, \tag{5.2.41}$$

for some constant $C''(M,\tau,t)$ depending on M, τ and t. Taking the limit when n goes to infinity gives the desired result.

The convergence of the functions $k_t^{H_n}(\cdot,0)$ in $\mathrm{L}^2(\mathbb{R}^d)$ plus the convergence of operators $e^{2tH_n}F(H_n)$ in the strong resolvent sense in (5.2.13) implies that

$$\lim_{n\to\infty} |f(H_n)(0,0) - f(H)(0,0)| = 0, \tag{5.2.42}$$

that is, the kernel $f(H_{P^\omega})(0,0)$ is continuous on $\hat{X}_D$. □

Because of the regularity of V_{D^ω}, for $z \in \mathbb{R}^d$ and $L > 0$, the expression

$$\frac{1}{L^d}\mathrm{tr}\, F(H_{D^\omega})\chi_{\Lambda_{z,L}} \tag{5.2.43}$$

is a well defined continuous positive linear functional on $\mathcal{C}_c(\mathbb{R})$, see e.g. [H08, Section 2.4.2]. By the Riesz representation theorem, it defines a measure that we denote by $n_{z,L}^{D^\omega}$,

$$\int_{\mathbb{R}} F(\lambda) dn_{z,L}^{D^\omega} = \frac{1}{L^d}\mathrm{tr}\, F(H_{D^\omega})\chi_{\Lambda_{z,L}}. \tag{5.2.44}$$

Definition 5.2.3. For $z \in \mathbb{R}^d$, $L > 0$, $E \in \mathbb{R}$ and D^ω a coloured Delone set we define the *finite-volume integrated density of states*

$$\nu_{z,L}^{D^\omega}(E) = \frac{1}{L^d}\mathrm{tr}\left(P_E(H_{D^\omega})\chi_{\Lambda_{z,L}}\right). \tag{5.2.45}$$

Note that $\nu_{z,L}^{D^\omega}(E)$, also called the *finite-volume eigenvalue counting function*, corresponds to the distribution function of the measure $n_{z,L}^{D^\omega}$, by the relation $n_{z,L}^{D^\omega}((\infty,E]) = \nu_{z,L}^{D^\omega}(E)$. When there is no place for confusion, we drop D^ω from the notation.

We have the following

Theorem 5.2.4. *Let D be a uniquely ergodic Delone set. The limit of the finite-volume integrated density of states $\nu_{z,L}^{D^\omega}(E)$ exists for $\mathbb{P}_D$-a.e. $\omega \in \Omega_D$ and for all $E \in \mathbb{R}$ except at most countably many. The limit is called the* integrated density of states *and is given by*

$$\nu(E) = \int_{\hat{X}_D} p_E(H_{D^\omega})(0,0)\, d\hat{\mu}(D^\omega) \tag{5.2.46}$$

Proof. Theorem 5.1.5 ensures that for any $F \in \mathcal{C}_c(\mathbb{R})$, $f(H_{D^\omega})$ the integral kernel associated to the operator $F(H_{D^\omega})$, and every $z \in \mathbb{R}^d$ we have

$$\lim_{L\to\infty} \int_{\mathbb{R}} F(\lambda) dn_{z,L}^{D^\omega} = \int_{\mathbb{R}} f(H_{D^\omega})(0,0)\, d\hat{\mu}(D^\omega). \tag{5.2.47}$$

for every $\tilde{\omega}$ in a set of full $\mathbb{P}_D$-probability $\tilde{\Omega}_D^F$ that might depend on F. The r.h.s. of (5.2.47) defines a measure n such that

$$\int_{\mathbb{R}} F(\lambda) dn = \lim_{L\to\infty} \int_{\mathbb{R}} F(\lambda) dn_{z,L}^{D^\omega}, \tag{5.2.48}$$

for every $\tilde{\omega} \in \tilde{\Omega}_D^F$. It remains to show that this holds in a set of full $\mathbb{P}_D$-probability *independently* of $F \in \mathcal{C}_c(\mathbb{R})$.

Take a dense subset C_0 in $\mathcal{C}_c(\mathbb{R})$ in the uniform topology (which is possible since $\mathcal{C}_c(\mathbb{R})$ is separable), and take

$$\Omega_0 = \bigcap_{F\in C_0} \Omega_D^F. \tag{5.2.49}$$

Since this is a countable intersection of sets of full measure, Ω_0 is a set of full measure in Ω_D. We have that (5.2.48) holds for every element in C_0. Now, take any $F \in \mathcal{C}_c(\mathbb{R})$ and a sequence $(F_m)_{m\in\mathbb{N}}$ in C_0 such that $F_m \to F$ in the uniform norm. Let us denote by $n(F)$, $n_{\omega,L}(F)$ the integral of F with respect to the measure n and $n_{z,L}^{D^\omega}$ respectively. By adding and subtracting terms we can show that, for every $\omega \in \Omega_0$,

$$|n_{\omega,L}(F) - n(F)| \le |n_{\omega,L}(F) - n_{\omega,L}(F_m)| + |n_{\omega,L}(F_m) - n(F_m)| + |n(F_m) - n(F)|. \tag{5.2.50}$$

Now, both $\operatorname{tr}(F(H_{D^\omega})\chi_\Lambda)$ and $\hat{\mu}(f(H_{D^\omega})(0,0))$ are continuous functions in F. Indeed, for the former, see e.g. [H08], as for the latter, is given by the integral over a space of measure 1 in (5.2.10), where the internal product is continuous. Taking the limit as m goes to ∞ we see that the first and third term in the r.h.s. tend to 0, as for the second term, we take $L \to \infty$ and use Theorem 5.1.5.

Thus, we have shown that the measure $n_{z,L}^{D^\omega}$ converges vaguely to n for $\mathbb{P}_D$-a.e. $\omega \in \Omega_D$.

Since H_{D^ω} is a lower semi-bounded operator, the supports of $n_{z,L}^{D^\omega}$ are uniformly bounded below, and so

$$\lim_{E\to-\infty} \lim_{L\to\infty} n_{z,L}^{D^\omega}((-\infty,E]) = 0 \tag{5.2.51}$$

then, by a criteria for convergence of distribution functions of measures [HLMW01, Proposition 4.3], for all $E \in \mathbb{R}$, except at most countably many, we have

$$\lim_{L\to\infty} \nu_{z,L}^{D^\omega}(E) = \lim_{L\to\infty} n_{z,L}^{D^\omega}((-\infty,E]) = n((-\infty,E]) = \nu(E). \tag{5.2.52}$$

This shows that the distribution function of $n_{z,L}^{D^\omega}$, that is, the finite-volume integrated density of states converges to the distribution function of the measure n, that is,

$$\nu(E) = \int_{\hat{X}_D} p_E(H_{D^\omega})(0,0)d\hat{\mu}(P^\omega). \tag{5.2.53}$$

□

We call n, the *density of states measure* (DOS). With Theorem 5.1.5 we can prove properties for $n(E)$ that hold for the DOS measure in the ergodic case. Under a condition on the finiteness of the colour pace $\mathbb{A}$, we have

Theorem 5.2.5. *Let D be a Delone set with finite local complexity, strict uniform pattern frequency and let $\mathbb{A}$ be a finite colour space. Let H_{D^ω} be its associated Delone–Anderson operator, where the single-site probability measure has an atom in each point of $\mathbb{A}$. Let n be its density of states measure. Then,*

$$\operatorname{supp} n = \sigma(H_{D^\omega}), \quad \forall \omega \in \Omega_D, \tag{5.2.54}$$

where $\operatorname{supp} n$ *is the topological support of the measure n. As a consequence, there exist subsets $\Sigma_{pp}, \Sigma_{ac}, \Sigma_{sc} \subset \mathbb{R}$ such that*

$$\sigma_\bullet(H_{D^\omega}) = \Sigma_\bullet, \quad \forall \omega \in \Omega_D, \tag{5.2.55}$$

where $\bullet = pp, ac, sc$.

Proof. This result was proven in the discrete case in [MR07, Lemma 6.3]. The same arguments follow in the continuous case taking care of approximating the function $\chi_I(H_{D^\omega})$ properly and using a sufficiently smooth cut-off function to obtain an orthogonal sequence of compactly supported functions. To illustrate where the geometric assumptions on D are needed, and for the convenience of the reader, we sketch the main steps of the proof.

Fix an open interval $I \subset \mathbb{R}$. We aim to show that if there exists $P \in X_D$ and $\omega \in \Omega_D$ such that for any $z \in \mathbb{R}^d$, $\operatorname{tr} \chi_I(H_{D^\omega})\chi_{\Lambda_{z,L}} > 0$, then $n(I) > 0$. Since D is uniquely ergodic, this implies the same statement for every $P \in X_D$ and $\mathbb{P}_P$-a.e. $\omega \in \Omega_P$, in the same lines of proof of [MR07, Lemma 6.3]. The converse statement follows from the definition of n (see also [MR07, Lemma 6.3]).

Let $E \in I$ be in the spectrum of H_{P^ω}, $\epsilon := \operatorname{dist}(E, \mathbb{R}\setminus I)$ and $\delta \in (0, \frac{\epsilon}{4})$. Take $\varphi \in \operatorname{Ran} \chi_{(E-\delta,E+\delta)}(H_{P^\omega}) \neq \emptyset$, then

$$\| (H_{P^\omega} - E)\, \varphi\| \leq \delta\|\varphi\|. \tag{5.2.56}$$

Define $\tilde{\varphi} := h_\delta(x)\varphi(x)$, where $h = h_\delta \in \mathcal{C}_c^\infty(\mathbb{R}^d)$ is such that $h_\delta \equiv 1$ on the cube $\Lambda_R \subset \mathbb{R}^d$ of sidelength R, and $\max\{\|\Delta h_\delta\|_\infty, 2\|\nabla h_\delta\|_\infty\} < \delta$. We have

$$\| (H_{P^\omega} - E)\, \tilde{\varphi}\| \leq 2\delta\|\tilde{\varphi}\|. \tag{5.2.57}$$

Let $K \subset \mathbb{R}^d$ be the smallest compact set such that $\operatorname{dist}(\mathbb{R}^d \setminus K, \operatorname{supp}\tilde{\varphi}) > \delta_u$, where δ_u is the radius of a ball containing the support of the single-site potential u in V_{P^ω}, see (4.1.2). Denote by $G_0 := P \cap K$ the pattern in P associated with K. Because of the strict uniform pattern frequency property, we know that G_0 is repeated in P infinitely many times, that is, we can find a sequence $(v_j)_{j\in\mathbb{N}_0}$, $j \in \mathbb{N}_0$, $v_j \in \mathbb{R}^d$, $v_0 = 0$, such that $G_j := P \cap (K + v_j) = G_0 + v_j$. For each G_j, there are at most finitely many patterns $G_{j'}$ that overlap with G_j. By dropping the overlapping terms, we extract a subsequence from (v_j), say $(v_{j'})$ for $j \in \mathbb{N}_0' \subset \mathbb{N}_0$, such that the patterns $(G_{j'})_{j'}$ are pairwise disjoint. Denote by $\tilde{\eta}_L(G_0)$ the number of translated copies of G_0 in Λ_L that do not overlap, that is, $\tilde{\eta}_L(G_0) = \sharp\{G_{j'} \subset P : \exists v_{j'} \in \Lambda_L \text{ s.t. } G_{j'} = G_0 + v_{j'}\}$. We still have

$$\lim_{L\to\infty} \frac{\tilde{\eta}_L(G_0)}{|\Lambda_L|} =: \eta > 0, \tag{5.2.58}$$

For the sake of notation, in the following we just write j for $j' \in \mathbb{N}_0'$. Consider the translated functions $\tilde{\varphi}_j := \tilde{\varphi}(x - v_j)$, for $j \in \mathbb{N}_0'$. We have that $\operatorname{supp}\tilde{\varphi}_j \subset K_j := K + v_j$ are pairwise disjoint, so $(\tilde{\varphi}_j)_{j\in\mathbb{N}_0'}$ is an orthogonal sequence.

Recall that we consider i.i.d. random variables and a compactly supported single-site potential in the definition of the Delone–Anderson operator. Moreover, by hypothesis, the colour space $\mathbb{A}$ is finite, and the single-site probability measure has an atom in each point of $\mathbb{A}$, i.e. $\mathbb{P}(a) > 0$, $\forall a \in \mathbb{A}$. Then, for the colouring of the patterns G_j we have that the events

$$A_j := \{\tilde{\omega} \in \Omega_P \,:\, \tilde{\omega}_{p+v_j} = \omega_p,\ \forall p \in G_j\}, \tag{5.2.59}$$

that represent a repetition of the colouring of G_0 in the pattern G_j, are all independent. Because of the $\mathbb{R}^d$-covariance of the probability measure and the assumption on the single-site measure, $\mathbb{P}_P(A_j) = \mathbb{P}_P(A_0) > 0$, for all $j \in \mathbb{N}_0'$. By the strong law of large numbers we have

$$\lim_{L\to\infty} \frac{1}{\tilde{\eta}_L(G_0)} \sum_{\substack{j\in\mathbb{N}_0' \\ K_j\subset\Lambda_L}} \chi_{A_j}(\tilde{\omega}) := \mathbb{P}_P(A_0) > 0 \tag{5.2.60}$$

for $\mathbb{P}_P$-a.e. $\omega \in \Omega_P$.

Now, take a function $F \in \mathcal{C}_c(\mathbb{R})$ such that $\operatorname{supp} F \subset I$, $F \leq \chi_I$ and

$$F|_{I(\epsilon)} = \chi_{I(\epsilon)}, \quad \text{for } I(\epsilon) = \left(E - \frac{\epsilon}{2}, E + \frac{\epsilon}{2}\right) \subset I. \tag{5.2.61}$$

We use F and Theorem 5.1.5 to obtain a lower bound on $n(I)$,

$$n(I) = \int_{\mathbb{R}} \chi_I \, dn \geq \int_{\mathbb{R}} F \, dn = \lim_{L\to\infty} \frac{\operatorname{tr} F(H_{P^\omega})\chi_{\Lambda_L}}{|\Lambda_L|}, \tag{5.2.62}$$

for $\mathbb{P}_P$-a.e. $\omega \in \Omega_P$. Now, take $\tilde{\omega} \in \Omega_0$, where Ω_0 is a set of full probability for which (5.2.60) and (5.2.62) hold. Then we can proceed as in [MR07] and obtain a lower bound for the trace of $F(H_{P^\omega})\chi_{\Lambda_L}$ by expanding it in the orthogonal sequence $(\tilde{\varphi}_j)_j$. We obtain,

$$\begin{aligned}
\operatorname{tr} F(H_{P^\omega})\chi_{\Lambda_L} &\geq \sum_{\substack{j\in\mathbb{N}_0' \\ K_j\subset\Lambda_L}} \frac{1}{\|\tilde{\varphi}\|^2}\langle\tilde{\varphi}_j, F(H_{P^{\tilde{\omega}}})\tilde{\varphi}_j\rangle \\
&\geq \sum_{\substack{j\in\mathbb{N}_0' \\ K_j\subset\Lambda_L}} \frac{\chi_{A_j}(\tilde{\omega})}{\|\tilde{\varphi}\|^2}\langle\tilde{\varphi}_j, F(H_{P^{\tilde{\omega}}})\tilde{\varphi}_j\rangle \qquad (5.2.63) \\
&\geq \sum_{\substack{j\in\mathbb{N}_0' \\ K_j\subset\Lambda_L}} \frac{\chi_{A_j}(\tilde{\omega})}{\|\tilde{\varphi}\|^2}\langle\tilde{\varphi}_j, \chi_{I(\epsilon)}(H_{P^{\tilde{\omega}}})\tilde{\varphi}_j\rangle \qquad (5.2.64)
\end{aligned}$$

where in the last inequality, we used that $\chi_{I(\epsilon)} \leq F$. The internal product in the r.h.s. of (5.2.64) can be bounded below as in [MR07, Eq. 6.13], that is,

$$\begin{aligned}
\langle\tilde{\varphi}_j, \chi_{I(\epsilon)}(H_{P^{\tilde{\omega}}})\tilde{\varphi}_j\rangle &= \|\tilde{\varphi}_j\|^2 - \|\chi_{\mathbb{R}\setminus I(\epsilon)}(H_{P^{\tilde{\omega}}})\tilde{\varphi}_j\|^2 \qquad (5.2.65) \\
&\geq \|\tilde{\varphi}_j\|^2 - \left(\frac{\epsilon}{2}\right)^{-2} \|(H_{P^{\tilde{\omega}}} - E)\tilde{\varphi}_j\|^2. \qquad (5.2.66)
\end{aligned}$$

For $\tilde{\omega} \in A_j$, we have

$$\|(H_{P^{\tilde{\omega}}} - E)\tilde{\varphi}_j\|^2 = \|(H_{P^\omega} - E)\varphi_j\|^2 < 4\delta^2\|\varphi\|^2 = 4\delta^2\|\tilde{\varphi}\|^2. \qquad (5.2.67)$$

Combining this, plus (5.2.60) and (5.2.58) in (5.2.63), and plugging an appropriate factor $\tilde{\eta}_L(G_0)$, we can separate the limit into a *pattern frequency* part times a *mean randomness* part:

$$\begin{aligned}
n(I) &\geq \left(1 - 16\frac{\delta^2}{\epsilon^2}\right) \liminf_{L\to\infty} \left(\frac{\tilde{\eta}_L(G_0)}{|\Lambda_L|} \cdot \frac{1}{\tilde{\eta}_L(G_0)} \sum_{\substack{j\in\mathbb{N}_0' \\ K_j\subset\Lambda_L}} \chi_{A_j}(\tilde{\omega}) \right) \qquad (5.2.68) \\
&\geq \left(1 - 16\frac{\delta^2}{\epsilon^2}\right) \eta\, \mathbb{P}_P(A_0) > 0.
\end{aligned}$$

□

Remark 5.2.6. In the case where the Delone dynamical system X_D is not uniquely ergodic,the integrated density of states $\nu(E)$ exists for μ-a.e. $P \in X_D$ and $\mathbb{P}_P$-a.e. $\omega \in \Omega_P$. In this case, the support of the density of states measure $\operatorname{supp} n = \sigma(H_{P^\omega})$ for μ-almost every $P \in X_D$ and $\mathbb{P}_P$-almost every $\omega \in \Omega_P$.

Now, define a measure $\tilde{n}_{z,L}^{D^\omega}$ as the finite-volume operator version of $n_{z,L}^{D^\omega}$: For any $F \in \mathcal{C}_c(\mathbb{R})$,

$$\int_{\mathbb{R}} F(\lambda) d\tilde{n}_{z,L}^{D^\omega} = \frac{1}{L^d} \operatorname{tr} F(H_{D^\omega}|_{\Lambda_{z,L}}) \qquad (5.2.69)$$

where $H_{D^\omega}|_{\Lambda_{z,L}}$ is the restriction of H_{D^ω} to the cube $\Lambda_{z,L}$ with self-adjoint boundary conditions. Its distribution function $\tilde{\nu}_{z,L}^{D^\omega}(E) = \tilde{n}_{z,L}^{D^\omega}((-\infty, E])$ is defined by

$$\tilde{\nu}_{z,L}^{D^\omega}(E) = \frac{1}{L^d} \operatorname{tr} P_E(H_{D^\omega}|_{\Lambda_{z,L}}) \qquad (5.2.70)$$

Since $H_{D^\omega}|_{\Lambda_{z,L}}$ has discrete spectrum, these quantities is well defined.

The behavior of limit of the distribution function of $\tilde{\nu}_{z,L}^{D^\omega}(E)$ as L goes to infinity and how it is related to $\nu_{z,L}^{D^\omega}(E)$ and $\nu(E)$ is given by the following

Proposition 5.2.7. *For every $E \in \mathbb{R}$, any $z \in \mathbb{R}^d$,*

$$\lim_{L\to\infty} \tilde{\nu}_{z,L}^{D^\omega}(E) = \nu(E), \quad \text{for } \mathbb{P}_D\text{-a.e. } \omega \in \Omega_D, \tag{5.2.71}$$

where $\nu(E)$ is the integrated density of states defined in Theorem 5.2.4.

Proof. By Theorem 5.2.4 we know that $n_{z,L}^{D^\omega}$ converges vaguely to n, for $\mathbb{P}_D$-a.e. $\omega \in \Omega_D$. On the other hand, [H08, Lemma 2.15] gives

$$\lim_{L\to\infty} \left| \int_{\mathbb{R}} F(\lambda) d\tilde{n}_{z,L} - \int_{\mathbb{R}} F(\lambda) dn_{z,L} \right| = 0, \tag{5.2.72}$$

for every $F \in \mathcal{C}_c(\mathbb{R})$. Therefore, $\tilde{n}_{z,L}$ converges vaguely to n as well. Since H_{D^ω} is a lower semi-bounded operator, the supports of $\tilde{n}_{z,L}$ are uniformly bounded below, and so

$$\lim_{E\to-\infty} \lim_{L\to\infty} n_{z,L}((-\infty, E]) = 0. \tag{5.2.73}$$

Then, by a criteria for convergence of distribution functions of measures [HLMW01, Proposition 4.3], for all $E \in \mathbb{R}$ we have

$$\lim_{L\to\infty} \tilde{\nu}_{z,L}(E) = \lim_{L\to\infty} \tilde{n}_{z,L}((-\infty, E]) = \nu(E) = n((-\infty, E]). \tag{5.2.74}$$

□

Since the convergence results for $n_{z,L}^{D^\omega}$, $\tilde{n}_{z,L}^{D^\omega}$ hold for any $z \in \mathbb{R}^d$ and for every $P \in X_D$, under the stated conditions on D, we simplify the notation and write n_L, $\tilde{n}_L$. We also use indistinctively the expression *a.s.* to denote "for $\mathbb{P}_D$-a.e. $\omega \in \Omega_D$" and Ω to denote Ω_D, when no confusion arises.

5.3 Lifshitz Tails

Recall that for H_{D^ω} the operator defined in (5.2.1) with $H_0 = -\Delta$ and V_{D^ω} defined by (5.2.3), satisfying (4.1.2), with $\operatorname{supp} \rho = [0, M]$, $M > 0$ and the additional assumption on the probability distribution ρ,

$$\rho([0, \epsilon]) \geq C_\rho \epsilon^\alpha, \quad \text{for some } C_\rho, \alpha > 0. \tag{5.3.1}$$

Then $\sigma(H_{D^\omega}) = [0, +\infty)$ a.s. and we have the following

Theorem 5.3.1. *Let $H_{D^\omega} = -\Delta + \lambda V_{D^\omega}$ with V_{D^ω} defined in (5.2.3) and D a Delone set with finite local complexity and strict uniform pattern frequency. Then, its integrated density of states $\nu(E)$ exhibits Lifshitz tails at the bottom of the spectrum.*

The proof, as in the case of a periodic Delone set, relies on the Neumann–Dirichlet bracketing of ν in terms of the finite-volume eigenvalue counting function $\tilde{\nu}_L(E)$ with Neumann and Dirichlet boundary conditions. It is enough to find upper and lower bounds for the functions $\tilde{\nu}_L(E)$ in average over $\hat{X}_D$ since, from Weyl asymptotics, for any $L \in \mathbb{N}$ and $P^\omega \in \hat{X}_D$ we have $|\tilde{\nu}_L(E)| \leq CE^{d/2} =: g(E)$, where $g \in L^1(\hat{X}_D, \hat{\mu})$ (see also property NE, Section 2.2). Then, recalling (5.2.5), we can apply Lebesgue's Dominated Convergence Theorem to the sequence $\tilde{\nu}_L(E)$ and obtain, for every $E \in \mathbb{R}$,

$$\lim_{L\to\infty} \hat{\mu}(\tilde{\nu}_L(E)) := \lim_{L\to\infty} \int_{\hat{X}_D} \tilde{\nu}_L(E) d\hat{\mu}(P^\omega) = \int_{\hat{X}_D} \nu(E) d\hat{\mu}(P^\omega) = \nu(E), \tag{5.3.2}$$

since $\hat{\mu}(\hat{X}_D) = 1$. This holds for any self adjoint boundary conditions on the cube Λ_L, in particular, for Neumann and Dirichlet boundary conditions. Next, we establish the Neumann–Dirichlet bracketing for ν.

5.3.1 Neumann–Dirichlet bracketing

Denote by $H_{D^\omega}|^N_{\Lambda_{z,L}}$ (resp. $H_{D^\omega}|^D_{\Lambda_{z,L}}$) the restriction of H_{D^ω} to a cube $\Lambda_{z,L}$ with Neumann (resp. Dirichlet) boundary conditions, and by $\tilde{\nu}^N_{z,L}(E)$ (resp. $\tilde{\nu}^D_{z,L}(E)$) its associated normalized eigenvalue counting function. If $z = 0$ we omit it from the subscript.

For a fixed $L \in \mathbb{N}$, take $K \in L\mathbb{N} \setminus \{1\}$ and consider the Følner sequence $\{\Lambda_{0,K}\}_{K \in L\mathbb{N}}$ of concentric cubes centered in 0, such that

$$\Lambda_{0,K} = \bigcup_{j \in \mathcal{J}} \Lambda_L(j), \tag{5.3.3}$$

for some index set $\mathcal{J} \subset \mathbb{Z}^d$, with $|\mathcal{J}| = (K/L)^d$. By the subadditivity of $\operatorname{tr} P_E(H_{D^\omega}|^N_{\Lambda_{0,K}})$ we have

$$\tilde{\nu}^N_K(E) = \frac{1}{|\Lambda_{0,K}|} \operatorname{tr} P_E(H_{D^\omega}|^N_{\Lambda_{0,K}}) \leq \frac{1}{|\Lambda_{0,K}|} \sum_{j\in\mathcal{J}} \operatorname{tr} P_E(H_{D^\omega}|^N_{\Lambda_{j,L}}) \tag{5.3.4}$$

$$= \frac{|\Lambda_{j,L}|}{|\Lambda_{0,K}|} \sum_{j\in\mathcal{J}} \frac{1}{|\Lambda_{j,L}|} \operatorname{tr} P_E(H_{P^\omega}|^N_{\Lambda_{j,L}}) \tag{5.3.5}$$

$$= \frac{1}{|\mathcal{J}|} \sum_{j\in\mathcal{J}} \tilde{\nu}^N_{j,L}(E). \tag{5.3.6}$$

Taking the integral with respect to the measure $\hat{\mu}$, and recalling that this measure is invariant with respect to translations in $\mathbb{R}^d$, in particular, translations in $\mathbb{Z}^d$, we obtain

$$\hat{\mu}(\tilde{\nu}^N_K(E)) \leq \frac{1}{|\mathcal{J}|} \sum_{j\in\mathcal{J}} \hat{\mu}\left(\tilde{\nu}^N_{j,L}(E)\right) \tag{5.3.7}$$

$$\leq \frac{1}{|\mathcal{J}|} \sum_{j\in\mathcal{J}} \hat{\mu}\left(\tilde{\nu}^N_{j,L}(E)\right) \tag{5.3.8}$$

$$\leq \hat{\mu}\left(\tilde{\nu}^N_L(E)\right). \tag{5.3.9}$$

Taking the limit as $K \to \infty$ and keeping L fixed, we see that

$$\nu(E) \le \hat{\mu}\left(\tilde{\nu}_L^N(E)\right). \tag{5.3.10}$$

In an analogous way, using the superadditivity of $\operatorname{tr} P_E(H_{D^\omega}|_{\Lambda_{0,K}}^D)$ we can prove

$$\hat{\mu}(\tilde{\nu}_K^D(E)) \ge \hat{\mu}\left(\tilde{\nu}_L^D(E)\right). \tag{5.3.11}$$

Taking the limit when $K \to \infty$ and keeping L fixed, gives

$$\nu(E) \ge \hat{\mu}\left(\tilde{\nu}_L^D(E)\right). \tag{5.3.12}$$

By (5.3.10) and (5.3.12),

$$\int_{X_D} \mathbb{E}_{\Omega_P}\left(\tilde{\nu}_L^D(E)\right) d\mu(P) \le \nu(E) \le \int_{X_D} \mathbb{E}_{\Omega_P}\left(\tilde{\nu}_L^N(E)\right) d\mu(P). \tag{5.3.13}$$

5.3.2 Proof of Theorem 5.3.1

Proof of Theorem 5.3.1. We obtain separately upper and lower bounds for $\nu(E)$ in (5.3.13), using an arbitrary $P^\omega \in \hat{X}_D$, following the arguments for the ergodic case. We will show that these bounds are uniform in $\hat{X}_D$, that is, they depend only on the parameters r and R which are common for all elements in $\hat{X}_D$.

i. Upper bound for $\mathbb{E}_{\Omega_P}\left(\tilde{\nu}_L^N(E)\right)$.

We proceed as in [K07], [KMe07]:

$$\mathbb{E}_{\Omega_P}\left(\tilde{\nu}_L^N(E)\right) = \frac{1}{|\Lambda_L|}\mathbb{E}_{\Omega_P}\left(\operatorname{tr} P_E(H_{P^\omega}|_{\Lambda_L}^N)\right) \tag{5.3.14}$$

$$\le \frac{1}{|\Lambda_L|}\mathbb{P}_P\left(E_1(H_{P^\omega}|_{\Lambda_L}^N) < E\right) \cdot \operatorname{tr} P_E(H_0|_{\Lambda_L}^N), \tag{5.3.15}$$

where $E_1(H_0|_{\Lambda_L}^N)$ denotes the ground state energy of $H_0|_{\Lambda_L}^N$. By Weyl asymptotics, there exist constants $C > 0$ and $C_{E,d} = CE^{d/2}$ such that the trace in the r.h.s. of the last equation is bounded by $C_{E,d}\,|\Lambda_L|$, therefore

$$\mathbb{E}_{\Omega_P}\left(\tilde{\nu}_L^N(E)\right) \le C_{E,d}\,\mathbb{P}_P\left(E_1(H_{P^\omega}|_{\Lambda_L}^N) < E\right). \tag{5.3.16}$$

To estimate the r.h.s., we will use (see, e.g. [RSIV]):

Lemma 5.3.2 (Temple's inequality). *Let H be a lower semi-bounded self-adjoint operator with discrete spectrum and denote by $E_1(H) \le E_2(H), \ldots$ its (increasing) eigenvalues, counting multiplicities. If $a \le E_2(H)$ and $\psi \in \mathcal{D}(H)$ with $\|\psi\|_2$ satisfying $\langle \psi, H\psi\rangle < a$, then*

$$E_1(H) \ge \langle \psi, H\psi\rangle - \frac{\langle \psi, H^2\psi\rangle - \langle \psi, H\psi\rangle^2}{a - \langle \psi, H\psi\rangle}. \tag{5.3.17}$$

Consider $\psi_0 = |\Lambda_L|^{-1/2}$, the ground state of $-\Delta^N_{\Lambda_L}$. Note that $E_1(-\Delta^N_{\Lambda_L}) = 0$, the second eigenvalue $E_2(\Delta^N_{\Lambda_L}) = cL^{-2}$ for some $c > 0$, and $E_2(-\Delta^N_{\Lambda_L}) \leq E_2(H_{P^\omega}|^N_{\Lambda_L})$. Define the random variables $\tilde{\omega}_p = \min\{\omega_p, \frac{c}{3L^2}\}$, for $p \in P \cap \Lambda_L$, and denote the corresponding Hamiltonian by $\tilde{H}_{P,\omega,L,N} = -\Delta^N_{\Lambda_L} + V_{P^{\tilde{\omega}},L}$, where we write $V_{P^{\tilde{\omega}},L} = V_{P^{\tilde{\omega}}}|_{\Lambda_L}$. Note that $\tilde{H}_{P,\omega,L,N}\psi_0 = V_{P^{\tilde{\omega}},L}\psi_0$ and, by the min-max principle, $E_1(\tilde{H}_{P,\omega,L,N}) \leq E_1(H_{P^\omega}|^N_{\Lambda_L})$.

Since $\operatorname{supp} u(x-p)$ are disjoint, we have

$$\langle \psi_0, V_{P^{\tilde{\omega}},L}\psi_0\rangle \leq \frac{c}{3L^2}, \text{ and } \langle \psi_0, V^2_{P^{\tilde{\omega}},L}\psi_0\rangle \leq \frac{c}{3L^2}\langle \psi_0, V_{P^{\tilde{\omega}},L}\psi_0\rangle. \tag{5.3.18}$$

We now apply Temple's inequality to $\tilde{H}_{P,\omega,L,N}$, taking ψ_0 and $a = E_2(-\Delta^N_{\Lambda_L}) = cL^{-2}$, which gives

$$E_1(\tilde{H}_{P,\omega,L,N}) \geq \langle \psi_0, \tilde{H}_{P,\omega,L,N}\psi_0\rangle - \frac{\langle \psi_0, V^2_{P^{\tilde{\omega}},L}\psi_0\rangle - \big(\langle \psi_0, V_{P^{\tilde{\omega}},L}\psi_0\rangle\big)^2}{cL^{-2} - \langle \psi_0, V_{P^{\tilde{\omega}},L}\psi_0\rangle} \tag{5.3.19}$$

$$\geq \langle \psi_0, V_{P^{\tilde{\omega}},L}\psi_0\rangle - \frac{\langle \psi_0, V_{P^{\tilde{\omega}},L}\psi_0\rangle (c/3)L^{-2}}{(c-c/3)L^{-2}} \tag{5.3.20}$$

$$\geq \frac{1}{2}\langle \psi_0, V_{P^{\tilde{\omega}},L}\psi_0\rangle = \frac{1}{2\,|\Lambda_L|}\sum_{p\in P\cap\Lambda_L}\int_{\Lambda_L}\tilde{\omega}_p u(x-p)dx \tag{5.3.21}$$

$$= \frac{C_{u,d}}{|\Lambda_L|}\sum_{p\in P\cap\Lambda_L}\tilde{\omega}_p, \tag{5.3.22}$$

where $C_{u,d} = u^-\epsilon_u^d/2$ (see (4.1.2)). Therefore,

$$\mathbb{E}_{\Omega_P}\left(\tilde{\nu}^N_L(E)\right) \leq C_{E,d}\,\mathbb{P}_P\left(\frac{1}{|\Lambda_L|}\sum_{p\in\Lambda_L\cap P}\tilde{\omega}_p \leq C_{u,d}E\right). \tag{5.3.23}$$

To estimate the r.h.s., we adapt [KMe07, Lemma 3.4] to Delone potentials in the following

Lemma 5.3.3. *For $L = \lfloor \beta E^{-1/2}\rfloor$ with $\beta > 0$ small and L large enough,*

$$\mathbb{P}_P\left(\frac{1}{|\Lambda_L|}\sum_{p\in\Lambda_L\cap P}\tilde{\omega}_p \leq C_{u,d}E\right) \leq e^{-C_1|\Lambda_L|}, \tag{5.3.24}$$

for some constant $C_1 = C_{\beta,u,d,R} > 0$ depending only on β, u, R and d, and therefore this bound is uniform for all $P^\omega \in \hat{X}_D$.

With this in hand, we can prove

$$\mathbb{E}_{\Omega_P}\left(\tilde{\nu}^N_L(E)\right) \leq C_{E,d}\,e^{-C_1L^d} \tag{5.3.25}$$

$$= C_{E,d}\,e^{-C_1\lfloor\beta E^{-1/2}\rfloor^d} \tag{5.3.26}$$

$$\leq C_{E,d}\,e^{-C_1' E^{-d/2}}. \tag{5.3.27}$$

Taking the integral with respect to the measure μ over X_D and recalling the last bound is uniform for all $P^\omega \in \hat{X}_D$ we get

$$\nu(E) \leq \hat{\mu}(\tilde{\nu}_L^N(E)) = \int_{X_D} \mathbb{E}_{\Omega_P}\left(\tilde{\nu}_L^N(E)\right) d\mu \tag{5.3.28}$$

$$\leq \int_{X_D} C_{E,d}\, e^{-C_1' E^{-d/2}} d\mu = C_{E,d}\, e^{-C_1' E^{-d/2}}, \tag{5.3.29}$$

where we used $\mu(X_D) = 1$, and so, we obtain the upper bound

$$\nu(E) \leq C e^{-C_1' E^{-d/2}} \quad \text{for } E \in [0,1]. \tag{5.3.30}$$

ii. Lower bound for $\mathbb{E}_{\Omega_P}\left(\tilde{\nu}_L^D(E)\right)$.

$$\mathbb{E}_{\Omega_P}\left(\tilde{\nu}_L^D(E)\right) = \frac{1}{|\Lambda_L|}\mathbb{E}_{\Omega_P}\left(\operatorname{tr} P_E(H_{P^\omega}|_{\Lambda_L}^D)\right) \tag{5.3.31}$$

$$\geq \frac{1}{|\Lambda_L|}\mathbb{P}_P\left(\operatorname{tr} P_E(H_{P^\omega}|_{\Lambda_L}^D \geq 1\right) \tag{5.3.32}$$

$$\geq \frac{1}{|\Lambda_L|}\mathbb{P}_P\left(E_1(H_{P^\omega}|_{\Lambda_L}^D) < E\right). \tag{5.3.33}$$

In order to minimize the r.h.s., we will take the ground state of $-\Delta_{\Lambda_L}^N$, $\psi_0 = |\Lambda_L|^{-1/2}$ multiplied by some smooth function which is zero in the boundary of Λ_L, so it satisfies Dirichlet boundary conditions, and therefore, is in the domain of the Dirichlet Laplacian. Consider a function $f \in \mathcal{C}_0^\infty(\Lambda_r)$ such that $f \equiv 1$ on $\operatorname{supp} u$, $0 \leq f \leq 1$ on $\Lambda_r \setminus \operatorname{supp} u$ and $|\Delta f| \leq c_0$, for some $c_0 > 0$. Set $f_L(x) = f(x/L)$ and denote by $\psi_L = \psi_0(x) \cdot f_L(x) \in \mathcal{D}(H_{P^\omega}|_{\Lambda_L}^D)$. Then

$$E_1(H_{P^\omega}|_{\Lambda_L}^D) \leq \langle \psi_L, H_{P^\omega}|_{\Lambda_L}^D \psi_L\rangle = \langle \psi_L, -\Delta|_{\Lambda_L}^D \psi_L\rangle + \langle \psi_L, V_{P^\omega,L}\psi_L\rangle \tag{5.3.34}$$

$$\leq \frac{c_0}{L^2} + \frac{c_{u,d}}{|\Lambda_L|}\sum_{p\in P\cap\Lambda_L} \omega_p, \tag{5.3.35}$$

where $c_{u,d} = u^+\delta_u^d$ (see (4.1.2)). Choosing $L = \lceil \sqrt{2c_0} E^{-1/2}\rceil$, we have

$$\mathbb{E}_{\Omega_P}\left(\tilde{\nu}_L^D(E)\right) \geq \frac{1}{|\Lambda_L|}\mathbb{P}_P\left(\frac{c_{u,d}}{|\Lambda_L|}\sum_{p\in\Lambda_L\cap P}\omega_p < E - c_0 L^{-2}\right) \tag{5.3.36}$$

$$\geq \frac{1}{|\Lambda_L|}\mathbb{P}_P\left(\frac{1}{|\Lambda_L|}\sum_{p\in\Lambda_L\cap P}\omega_p < \frac{E}{2c_{u,d}}\right) \tag{5.3.37}$$

$$\geq \frac{1}{|\Lambda_L|}\mathbb{P}_P\left(\omega_0 < \frac{r^d}{2c_{u,d}}E\right)^{|\Lambda_L\cap P|} \tag{5.3.38}$$

$$\geq \frac{1}{|\Lambda_L|}\mathbb{P}_P\left(\omega_0 < C_{r,u,d}E\right)^{C_{r,d}|\Lambda_L|}, \tag{5.3.39}$$

where, in the last two inequalities we took into account that $|\Lambda_L \cap P| \le C_{r,d}|\Lambda_L|$, with $C_{r,d} = r^{-d}$, since $P \in \mathcal{P}_r(\mathbb{R}^d)$. Recalling (5.3.1) and the choice of L, we obtain

$$\mathbb{E}_{\Omega_P}\left(\tilde{\nu}_L^D(E)\right) \ge \frac{1}{L^d} C_\mu (C_{c0,u,r,d}E)^{\alpha C_{r,d} L^d} \tag{5.3.40}$$

$$\ge C_{\mu,c_0,r,d} E^{d/2} e^{-C_{c_0,\alpha,r,d}E^{-d/2}|\log E|}, \tag{5.3.41}$$

for E small enough.

Since the last bound is uniform for all $P^\omega \in \hat{X}_D$, we can take the integral with respect to the measure μ as in the case of the upper bound, and get

$$\nu(E) \ge C_{\mu,\beta,r,d} E^{d/2} e^{-C_{\beta,\alpha,r,d}E^{-d/2}|\log E|}, \tag{5.3.42}$$

which concludes the proof of the lower bound for the IDS, proving Lifshitz tails for energies E near 0. □

Proof of Lemma 5.3.3 . We follow the proof of [K07, Lemma 6.4]

$$\mathbb{P}_P\left(\frac{1}{|\Lambda_L|}\sum_{\gamma\in\Lambda_L\cap P}\tilde{\omega}_\gamma \le C_{u,d}E\right) \le \mathbb{P}_P\left(\frac{1}{|\Lambda_L|}\sum_{\gamma\in\Lambda_L\cap P}\tilde{\omega}_\gamma \le C_{u,d}\beta^2 L^{-2}\right) \tag{5.3.43}$$

$$\le \mathbb{P}_P\left(|\{\gamma \in \Lambda_L \cap P : \quad \tilde{\omega}_\gamma < \frac{c}{6}L^{-2}\}| \ge \left(1 - \frac{6\beta^2 C_{u,d}R^d}{c}\right)|P \cap \Lambda_L|\right). \tag{5.3.44}$$

(5.3.45)

This, because $\tilde{\omega}_\gamma \le \frac{c}{3L^2}$, $|\gamma \in \Lambda_L \cap P| \le \frac{|\Lambda_L|}{r^d}$ and so, with β small enough,

$$\left|\{\gamma \in \Lambda_L \cap P : \quad \tilde{\omega}_\gamma \le \frac{c}{6}L^{-2}\}\right| \le \left(1 - \frac{6\beta^2 C_{u,d}R^d}{c}\right)|P \cap \Lambda_L| \tag{5.3.46}$$

implies

$$\left|\{\gamma \in \Lambda_L \cap P : \quad \tilde{\omega}_\gamma > \frac{c}{6L^2}\}\right| \ge \frac{6\beta^2 C_{u,d}R^d}{c}\,|P \cap \Lambda_L|\,, \tag{5.3.47}$$

in which case,

$$\frac{1}{|\Lambda_L|}\sum_{\gamma\in\Lambda_L\cap P}\tilde{\omega}_\gamma > \frac{1}{|\Lambda_L|}\frac{c}{6L^2}\frac{6\beta^2 C_{u,d}R^d}{c}\,|P \cap \Lambda_L| = C_{u,d}\beta^2 L^{-2}, \tag{5.3.48}$$

where we used the fact that $|P \cap \Lambda_L| \ge |\Lambda_L|/R^d$ and choose β small such that $6\beta^2 C_{u,d}R^d/c < 1$. Now fix some $\delta > 0$ small and set $q := \mathbb{P}_P(\tilde{\omega}_\gamma < \delta) < 1$ and define new random variables ξ_γ as $\xi_\gamma = 1$ if $\tilde{\omega}_\gamma < \delta$, $\xi_\gamma = 0$ otherwise. The random variables $\{\xi_\gamma\}$ are i.i.d. and $\mathbb{E}_{\Omega_P}(\xi_\gamma) = q$.

Define $\rho = 1 - \frac{6\beta^2 C_{u,d}R^d}{c}$. By taking β small enough we can ensure that $q < \rho < 1$. Then, for L big enough,

$$\mathbb{P}_P\left(\left|\{\gamma\in\Lambda_L\cap P:\quad \tilde{\omega}_\gamma<\frac{2c}{3}L^{-2}\}\right|\geq\rho\,|P\cap\Lambda_L|\right)\leq\mathbb{P}_P\left(|\{\gamma\in\Lambda_L\cap P:\tilde{\omega}_\gamma<\delta\}|\geq\rho\,|P\cap\Lambda_L|\right) \tag{5.3.49}$$

$$\leq\mathbb{P}_P\left(\frac{1}{|P\cap\Lambda_L|}\sum_{\gamma\in\Lambda_L\cap P}\xi_\gamma\geq\rho\right). \tag{5.3.50}$$

$$\tag{5.3.51}$$

The last line is a large deviations problem and can be estimated in a standard way, using the inequality

$$\mathbb{P}_P(X>a)\leq e^{-ta}\mathbb{E}_{\Omega_P}(e^{tX})\text{ for all }t\geq 0. \tag{5.3.52}$$

So we obtain

$$\mathbb{P}_P\left(\frac{1}{|\Lambda_L\cap P|}\sum_{\gamma\in\Lambda_L\cap P}\xi_\gamma\geq\rho\right)\leq e^{-t\rho|\Lambda_L\cap P|}\mathbb{E}_{\Omega_P}\left(e^{t\sum_\gamma\xi_\gamma}\right) \tag{5.3.53}$$

$$\leq e^{-t\rho|\Lambda_L\cap P|}\prod_\gamma\mathbb{E}_{\Omega_P}\left(e^{t\xi_\gamma}\right) \tag{5.3.54}$$

$$\leq e^{-t\rho|\Lambda_L\cap P|}\mathbb{E}_{\Omega_P}\left(e^{t\xi_0}\right)^{|\Lambda_L\cap P|} \tag{5.3.55}$$

$$\leq e^{-t\rho|\Lambda_L\cap P|+|\Lambda_L\cap P|\ln\mathbb{E}_{\Omega_P}\left(e^{t\xi_0}\right)} \tag{5.3.56}$$

$$\leq e^{-|\Lambda_L\cap P|(t\rho-\ln\mathbb{E}_{\Omega_P}\left(e^{t\xi_0}\right))}. \tag{5.3.57}$$

Set $f(t):=t\rho-\ln\mathbb{E}_{\Omega_P}\left(e^{t\xi_0}\right)$. Then $f'(t)=\rho-\frac{\mathbb{E}_{\Omega_P}\left(\xi_0e^{t\xi_0}\right)}{\mathbb{E}_{\Omega_P}\left(e^{t\xi_0}\right)}$, and in particular $f'(0)=\rho-q>0$. Since $f(0)=0$, there is a $t_0>0$ such that $f(t_0)>0$. We then have

$$\mathbb{P}_P\left(\frac{1}{|\Lambda_L\cap P|}\sum_{\gamma\in\Lambda_L\cap P}\xi_\gamma\geq\rho\right)\leq e^{-|\Lambda_L\cap P|f(t_0)} \tag{5.3.58}$$

$$\leq e^{-\frac{L^d}{R^d}f(t_0)}=e^{-CL^d}. \tag{5.3.59}$$

where we use the fact that $|\Lambda_L\cap P|\geq\frac{L^d}{R^d}$ for $L>R$ and $C=f(t_0)/R^d$ is a constant that depends on β,u,R, and d.

□

5.4 Example of a Delone operator with no IDS

Consider two periodic potentials V_1 and V_2 of periods q_1 and q_2, respectively, with $q_1\neq q_2$, defined by

$$V_1(x) = \sum_{\gamma \in (q_1\mathbb{Z})^d} u(x-\gamma), \tag{5.4.1}$$

$$V_2(x) = \sum_{\gamma \in (q_2\mathbb{Z})^d} u(x-\gamma), \tag{5.4.2}$$

where u is a smooth function with $\operatorname{supp} u \subset \Lambda_{\min\{q_1,q_2\}/2}$ (that is, in both cases the supports of $u(x-\gamma)$ are disjoint. Now, consider a sequence of (open) cubes centered in the origin, $\{\Lambda_{L_k}\}_{k\in\mathbb{N}}$, with $L_{k+1} = L_k^\alpha$, $\alpha > 1$. Define $\mathbb{N}_e = \{2k : k \in \mathbb{N}\}$, $\mathbb{N}_o = \{2k-1 : k \in \mathbb{N}\}$, and consider the following covering of $\mathbb{R}^d$,

$$\mathbb{R}^d = \bigcup_{k=1}^{\infty} A_k, \quad A_k = \overline{\Lambda}_{L_k} \setminus \Lambda_{L_{k-1}}, \, \Lambda_{L_0} = \emptyset. \tag{5.4.3}$$

Let $H = H_0 + V$ be a selfadjoint operator acting on $\mathrm{L}^2(\mathbb{R}^d)$, where H_0 is lower semi-bounded and

$$V = \sum_{k\in\mathbb{N}_e} V_1 \chi_{A_k} + \sum_{k\in\mathbb{N}_o} V_2 \chi_{A_k}. \tag{5.4.4}$$

Notice that H can also be seen as a Delone operator,

$$H = H_0 + \sum_{\gamma\in D} u(x-\gamma), \tag{5.4.5}$$

where the Delone set D is defined by (see Fig. 5.1)

$$D = \left(\bigcup_{k\in\mathbb{N}_e} q_1\mathbb{Z}^d \cap A_k \right) \cup \left(\bigcup_{k\in\mathbb{N}_o} q_2\mathbb{Z}^d \cap A_k \right). \tag{5.4.6}$$

Proposition 5.4.1. *Let ν_{H,L_k} be the finite-volume integrated density of states (IDS) for the operator H as in Definition 5.2.3. The limit of ν_{H,L_k} when k tends to infinity does not exist.*

Proof. Without loss of generality, fix $k \in \mathbb{N}_e$. We write $H_1 = H_0 + V_1$ and $H_2 = H_0 + V_2$, for which $\nu_{H_1,L}$ and ν_{H_2,L_k} denote the respective finite-volume IDS restricted to a cube of side L_k. For a fixed $E \in \mathbb{R}$, consider the interval $(-\infty, E]$. Since we deal with lower semi-bounded operators, $\chi_{(-\infty,E]}(H) = \chi_I(H)$, where $I = (E_0, E) \subset \mathbb{R}$ is bounded. Consider a function $f \in \mathcal{C}_c(\mathbb{R})$ such that $I \subset \operatorname{supp} f$. We have

$$\begin{aligned}
\operatorname{tr} f(H)\chi_{\Lambda_{L_k}} &= \operatorname{tr} f(H) \left(\chi_{A_k} + \chi_{\Lambda_{L_{k-1}}} \right) && (5.4.7)\\
&= \operatorname{tr} f(H)\chi_{A_k} \pm \operatorname{tr} f(H_1)\chi_{\Lambda_{L_{k-1}}} + \operatorname{tr} f(H)\chi_{\Lambda_{L_{k-1}}} && (5.4.8)\\
&= \operatorname{tr} \left(f(H)\chi_{A_k} + f(H_1)\chi_{\Lambda_{L_{k-1}}} \right) + \operatorname{tr} \left(f(H)\chi_{\Lambda_{L_{k-1}}} - f(H_1)\chi_{\Lambda_{L_{k-1}}} \right) && (5.4.9)\\
&= \operatorname{tr} \left(f(H)\chi_{A_k} - f(H_{A_k})\chi_{A_k} \right) + \operatorname{tr} \left(f(H_{A_k})\chi_{A_k} + f(H_1)\chi_{\Lambda_{L_{k-1}}} \right) && (5.4.10)\\
&\quad + g(H, H_1, \Lambda_{L_{k-1}}), && (5.4.11)\\
& && (5.4.12)
\end{aligned}$$

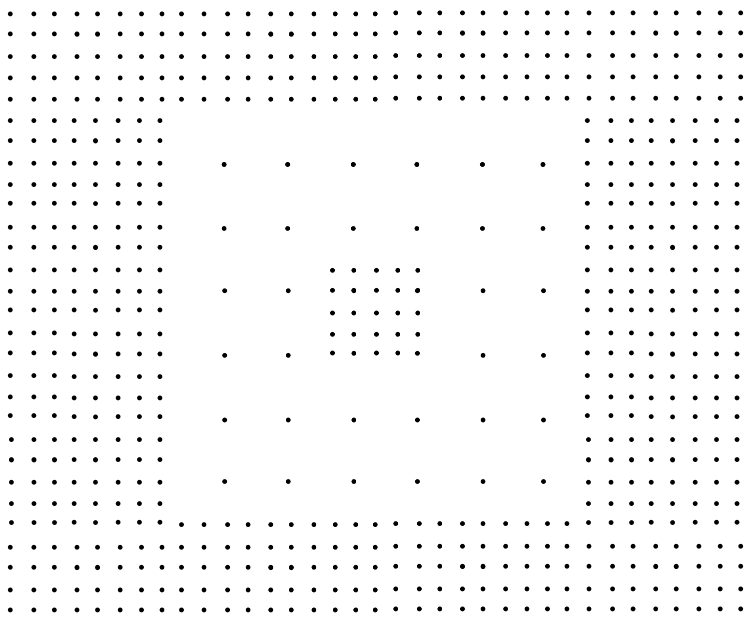

Figure 5.1: The Delone set D.

where H_{A_k} denotes the restriction of H to the set A_k, and

$$g(H, H_1, \Lambda_{L_{k-1}}) := \operatorname{tr}\left(f(H)\chi_{\Lambda_{L_{k-1}}} - f(H_1)\chi_{\Lambda_{L_{k-1}}}\right). \tag{5.4.13}$$

By the definition of V, for k even, we have $H_{A_k} = H_1|_{A_k}$, therefore

$$\begin{aligned} \operatorname{tr} f(H)\chi_{\Lambda_{L_k}} &= \operatorname{tr}\left(f(H) - f(H_{A_k})\right)\chi_{A_k} + \operatorname{tr}\left(f(H_1|_{A_k})\chi_{A_k} + f(H_1)\chi_{\Lambda_{L_{k-1}}}\right) && (5.4.14)\\ &\quad + g(H, H_1, \Lambda_{L_{k-1}}) && (5.4.15)\\ &= g(H, H_{A_k}, A_k) + \operatorname{tr}\left(f(H_1|_{A_k})\chi_{A_k} - f(H_1)\chi_{A_k}\right) + \operatorname{tr}\left(f(H_1)\chi_{A_k} + f(H_1)\chi_{\Lambda_{L_{k-1}}}\right) && (5.4.16)\\ &\quad + g(H, H_1, \Lambda_{L_{k-1}}) && (5.4.17)\\ &= \operatorname{tr} f(H_1)\chi_{\Lambda_{L_k}} + g(H, H_1, \Lambda_{L_{k-1}}) + g(H, H_{A_k}, A_k) + g(H_1|_{A_k}, H_1, A_k), && (5.4.18)\\ & && (5.4.19) \end{aligned}$$

where $g(H, H_{A_k}, A_k)$ and $g(H_1|_{A_k}, H_1, A_k)$ are defined analogously as (5.4.13). Then

$$\begin{aligned} \frac{1}{|\Lambda_{L_k}|}\operatorname{tr} f(H)\chi_{\Lambda_{L_k}} &= \frac{1}{|\Lambda_{L_k}|}\operatorname{tr} f(H_1)\chi_{\Lambda_{L_k}} + \mathcal{O}_1 + \mathcal{O}_2 + \mathcal{O}_3 \\ &= \int_{\mathbb{R}} f(\lambda)\mathrm{d}n_{H_1,L_k} + \mathcal{O}_1 + \mathcal{O}_2 + \mathcal{O}_3, && (5.4.20)\\ & && (5.4.21) \end{aligned}$$

where n_{H_1,L_k} denotes the finite-volume density of states measure associated to the operator H_1

and

$$\mathcal{O}_1 = \frac{1}{|\Lambda_{L_k}|} g(H, H_1, \Lambda_{L_{k-1}}), \tag{5.4.22}$$

$$\mathcal{O}_2 = \frac{1}{|\Lambda_{L_k}|} g(H, H_{A_k}, A_k), \tag{5.4.23}$$

$$\mathcal{O}_3 = \frac{1}{|\Lambda_{L_k}|} g(H_1|_{A_k}, H_1, A_k). \tag{5.4.24}$$

(5.4.25)

We will show that the terms $\mathcal{O}_1$, $\mathcal{O}_2$ and $\mathcal{O}_3$ tend to 0 as $k \to \infty$ in $\mathbb{N}_e$. By the same arguments that prove Proposition 5.2.7, namely (5.2.72) , the terms $\mathcal{O}_2$ and $\mathcal{O}_3$ converge to 0, while for $\mathcal{O}_1$ we have

$$\mathcal{O}_1 = \frac{1}{|\Lambda_{L_k}|} \operatorname{tr} \Big(f(H|_{\Lambda_{L_{k-1}}}) - f(H_1|_{\Lambda_{L_{k-1}}}) \Big) \chi_{\Lambda_{L_{k-1}}} + \frac{\left|\Lambda_{L_{k-1}}\right|}{|\Lambda_{L_k}|} (\mathcal{O}_1^1 + \mathcal{O}_2^1), \tag{5.4.26}$$

where

$$\mathcal{O}_1^1 = \frac{1}{\left|\Lambda_{L_{k-1}}\right|} g(H, H_{\Lambda_{L_{k-1}}}, \Lambda_{L_{k-1}}), \quad \mathcal{O}_2^1 = \frac{1}{\left|\Lambda_{L_{k-1}}\right|} g(H_1|_{\Lambda_{L_{k-1}}}, H_1, \Lambda_{L_{k-1}}). \tag{5.4.27}$$

Again, by (5.2.72), terms $\mathcal{O}_1^1$ and $\mathcal{O}_2^1$ tend to 0 as $k \to \infty$. By Weyl's inequality, and since $\chi_{\Lambda_{L_{k-1}}}$ is bounded, we have

$$\operatorname{tr} f(H|_{\Lambda_{L_{k-1}}}) \chi_{\Lambda_{L_{k-1}}} \leq C_{E,d} \left|\Lambda_{L_{k-1}}\right|, \quad \text{and} \quad \operatorname{tr} f(H_1|_{\Lambda_{L_{k-1}}}) \chi_{\Lambda_{L_{k-1}}} \leq C_{E,d} \left|\Lambda_{L_{k-1}}\right| \tag{5.4.28}$$

therefore, $\mathcal{O}_1$ tends to 0 as $k \to \infty$.

Then, we have from (5.4.20) that if $k \in \mathbb{N}_e$,

$$\lim_{k \to \infty} \nu_{H,L_k}(E) = \nu_{H_1}(E) \tag{5.4.29}$$

in an analogous way one proves that if $k \in \mathbb{N}_o$

$$\lim_{k \to \infty} \nu_{H,L_k}(E) = \nu_{H_2}(E) \tag{5.4.30}$$

Since $q_1 \neq q_2$, $\nu_{H,L_k}(E)$ does not have a limit when $k \to \infty$. □

Proposition 5.4.2. *The Delone set D defined in* (5.4.6) *does not have the* uniform pattern frequency *property* (5.1.19). *Therefore, the Delone dynamical system X_D is not uniquely ergodic.*

Proof. Without loss of generality, assume $q_1, q_2 \geq 1$ and consider the covering of $\mathbb{R}^d$ defined in (5.4.3), the sequence Λ_{L_k} and fix $k \in \mathbb{N}_e$. Take the pattern $Q = \{(0, 0, ...)\}$ with support $B_{1/2}(0)$ consisting in only one point, the origin in $\mathbb{R}^d$. Since D consists of translations of Q such that the translations of its support are disjoint, we have that the number of $B_{q_2}(0)$-patterns in Λ_{L_k}

that are translations of Q by an element of Λ_{L_k} is given by

$$\sharp\left\{\tilde{Q} \subset D : \exists y \in \Lambda_{L_k} \text{ s. t. } y + \tilde{Q} = Q\right\} = \sharp\left\{\tilde{Q} \subset D : \exists y \in A_k \cup \Lambda_{L_{k-1}} \text{ s. t. } y + \tilde{Q} = Q\right\} \tag{5.4.31}$$

$$= \sharp\left\{\tilde{Q} \subset D : \exists y \in A_k \text{ s. t. } y + \tilde{Q} = Q\right\} + \tag{5.4.32}$$

$$\sharp\left\{\tilde{Q} \subset D : \exists y \in \Lambda_{L_{k-1}} \text{ s. t. } y + \tilde{Q} = Q\right\} \tag{5.4.33}$$

$$= \sharp\{A_k \cap q_1\mathbb{Z}^d\} + \sharp\{\Lambda_{L_{k-1}} \cap \mathbb{Z}^d\} \tag{5.4.34}$$

$$= q_1^{-d}\,|A_k| + \left|\Lambda_{L_{k-1}}\right| \tag{5.4.35}$$

Then, we have, by the definition of the local pattern frequency of Q (5.1.19),

$$\eta_k(Q) := \frac{\eta_{0,L_k}(Q)}{|\Lambda_{L_k}|} = \frac{1}{|\Lambda_{L_k}|}\left(q_1^{-d}\,|A_k| + \left|\Lambda_{L_{k-1}}\right|\right) \tag{5.4.36}$$

Analogously, if we proceed in the same way for $k \in \mathbb{N}_o$, we obtain

$$\eta_k(Q) := \frac{\eta_{0,L_k}(Q)}{|\Lambda_{L_k}|} = \frac{1}{|\Lambda_{L_k}|}\left(q_2^{-d}\,|A_k| + \left|\Lambda_{L_{k-1}}\right|\right) \tag{5.4.37}$$

Recalling that $A_k = \Lambda_{L_k} \setminus \Lambda_{L_{k-1}}$ and $L_{k+1} = L_k^{\alpha}$ with $0 < \alpha < 1$, we see that by taking a subsequence of $(\eta_k(Q))$ with $k \in \mathbb{N}_e$, $(\eta_k(Q))$ converges to q_1^{-d}, while taking the sequence with $k \in \mathbb{N}_o$, it converges to q_2^{-d}.

By [MR12, Proposition 2.32] (see also [LS03, Theorem 1.7], [LeMoSo02, Theorem 2.7] which apply to our setting) the fact that D does not have the uniform patter frequency property implies that X_D is not uniquely ergodic. □

Appendix A

Contents

A.1 Quantitative unique continuation principle in arbitrary dimension

In this section we recall a quantitative unique continuation principle for eigenfunctions of a bounded perturbation of the Laplacian in arbitrary dimension, proved in [RMV12].

Let $\Lambda_L \subset \mathbb{R}^d$ be a cube of sidelength L centered in an arbitrary point of $\mathbb{R}^d$ (the following results are uniform with respect to the center of the box). Let $H_0 = -\Delta + V_0$ on $\mathrm{L}^2(\mathbb{R}^d)$, where V_0 is a bounded potential and denote by $H_{0,L}$, $V_{0,L}$ their respective restrictions to Λ_L, with Dirichlet boundary conditions and domain $\mathcal{D}(H_{0,L}) = H_0^2(\Lambda_L)$. We restate [RMV12, Theorem 2.1] for the particular case of eigenfunctions of $H_{0,L}$ on a (k, K)-Delone set.

Theorem A.1.1. *Let $I \subset \mathbb{R}$ be a compact interval and ψ an eigenfunction of $H_{0,L}$ associated to the eigenvalue $E \in I$, that is,*

$$H_L\psi := (-\Delta_L + V_{0,L} - E)\psi = 0 \quad \text{on } \Lambda_L. \tag{A.1.1}$$

Fix $K \in (0,\infty), \delta \in (0, K/4]$. There exists a constant $C_{sfUC,d} = C_{sfUC,d}(d, K_{V_0,E}, K, \delta) > 0$ such that, for any $L \in K\mathbb{N}$, any sequence

$$\{z_j\}_{j\in K\mathbb{Z}^d} \text{ in } \mathbb{R}^d \text{ such that } \quad B(z_j,\delta) \subset \Lambda_K(j), \quad \text{for all } j \in K\mathbb{Z}^d \tag{A.1.2}$$

we have

$$\sum_{j\in\tilde{\Lambda}_L} \|\psi\|^2_{B(z_j,\delta)} \geq C_{sfUC,d}\|\psi\|^2_{\Lambda_L} \tag{A.1.3}$$

where $K_{V_0,E} = \|V_0\|_\infty + \sup I$ *and* $\tilde{\Lambda}_L = \Lambda_L \cap K\mathbb{Z}^d$. *The dependence on* K *in the constant* $C_{sfUC,d}$ *is given explicitly by*

$$C_{sfUC,d} = c_1 K^{-c_2 K^{4/3} + c_3}, \tag{A.1.4}$$

for some constants c_1, c_2, c_3 *depending on* $d, \delta, K_{V_0,E}$.

Remark A.1.2.

i. In our applications, the set $\{z_j\}_{j\in K\mathbb{Z}^d}$ corresponds to a (k, K)-Delone set. We often write $D := \{z_j\}_{j\in K\mathbb{Z}^d}$.

ii. Since we are interested in a lower bound for (A.1.3), a given Delone set can always be reduced, by removing points, to a (k, K)-Delone set satisfying (A.1.2).

Before proceeding with the proof of Theorem A.1.1, we recall [RMV12, Theorem 3.1] (see also [GK11, Corollary A.2])

Theorem A.1.3. *Let* $K_V, D_0, R, \beta \in [0, \infty), \delta \in (0, 1]$. *There exists a constant* $C_{qUC} = C_{qUC}(d, K_V, D_0, R, \delta, \beta) > 0$ *such that, for any* $G \subset \mathbb{R}^d$ *open, any* $x \in G$*, any* $\Theta \subset G$ *measurable, satisfying the geometric conditions*

$$\operatorname{diam}\Theta \le R := \operatorname{dist}(x, \Theta), \quad D_0 < 12R, \quad \text{and} \quad B(x, 12R + 2D_0) \subset G,$$

and any measurable $V : G \to [-K_V, K_V]$ *and real-valued* $\psi \in W^{2,2}(G)$ *satisfying the differential inequality*

$$|\Delta\psi| \le |V\psi| \quad \text{a.e. on } \in G \quad \text{as well as} \quad \frac{\|\psi\|_G^2}{\|\psi\|_\Theta^2} \le \beta \tag{A.1.5}$$

we have

$$\|\psi\|^2_{B(x,\delta)} \ge C_{qUC} \|\psi\|^2_\Theta \tag{A.1.6}$$

Corollary A.1.4. *Let* $K_V, R, \beta \in [0, \infty), \delta \in (0, 1]$. *Let* $G \subset \mathbb{R}^d$ *open,* $x \in G$*,* $\Theta \subset G$ *measurable, satisfy the geometric conditions*

$$R \in \left[\sqrt{d}, 2\lceil\sqrt{d}\rceil\right], \ \operatorname{diam}\Theta \le R := \operatorname{dist}(x, \Theta) \quad \text{and} \quad B(x, 14R) \subset G, \tag{A.1.7}$$

and $V : G \to [-K_V, K_V]$ *measurable,* $\psi \in W^{2,2}(G)$ *real-valued, satisfy*

$$|\Delta\psi| \le |V\psi| \quad \text{a.e. on } \in G \quad \text{and} \quad \frac{\|\psi\|_G^2}{\|\psi\|_\Theta^2} \le \beta \tag{A.1.8}$$

Then there exists a constant $C = C(d) \in (1, \infty)$ *depending only on the dimension, such that*

$$\|\psi\|^2_{B(0,\delta)} \ge C_{qUC} \|\psi\|^2_\Theta \quad \text{where} \quad C_{qUC} := \left(\frac{\delta}{C}\right)^{C + C K_V^{2/3} + \ln\beta} \tag{A.1.9}$$

Remark A.1.5. More precisely, we have the following lower bound (see [RMV12, Remark 3.3-(b)])

$$C_{UCP}(d, K_{V_0,E}, R, \delta, \tau) \ge C_{C,d,V_0,E} \frac{\delta^4}{R^2} \left(\frac{\delta}{48R}\right)^{2\alpha}, \tag{A.1.10}$$

where $C_{C,d,V_0,E}$ involves constants that come from the Carleman estimate [BoK05, GK11, RMV12] depending only on the dimension d and $K_{V_0,E}$. By choosing $D_0 = R$ in Theorem A.1.3, the exponent α can be taken as

$$\alpha = \max\{C_a, (C_b K^2_{V_0,E})^{4/3} R^{4/3}, \ln(C_c(1 + K^2_{V_0,E})\,\tau)\}, \tag{A.1.11}$$

where C_a, C_b, C_c are constants coming from the Carleman estimate.

Without loss of generality, in the following we assume $D_0 = R \geq 1$.

Proof of Theorem A.1.1. The proof of [RMV12, Theorem] is stated for the case where the sequence $\{z_j\}_{j\in K\mathbb{Z}^d}$ follows the configuration of a (k,K)-Delone set with $K=1$ (see (A.1.2)). In order to see how the constant C_{sfUC} depends on the parameter K of the Delone set, in the following we reproduce the proof from [RMV12], in the case of eigenfunctions, writing explicitly the constant K, and without loss of generality, we assume $K \geq 1$.

A.1.1 Extension of the eigenfunction equation

We want to apply the quantitative unique continuation principle which requires among its geometric conditions a certain security distance to the boundary of the set where the eigenfunction equation is satisfied. This is not true for the solution ψ defined on the original cube Λ_L, therefore we will extend it to a larger set in such a way that the extension $\tilde{\psi}$ still satisfies an eigenfunction equation.

We restrict ourselves to Dirichlet boundary conditions on $\partial\Lambda_L$ (for periodic b.c. see [RMV12]). We make use of the same construction as in Corollary A.2 in [GK11].

The idea is to extend the potential V by symmetric reflections w.r.t. to hypersurfaces forming the boundaries of the cube Λ_L and afterwards extend the function ψ by antisymmetric reflections w.r.t. to hypersurfaces forming the boundaries of the cube Λ_L. Let us describe this more precisely.

For convenience we shift the coordinate system such that

$$\Lambda_L = \{x \in \mathbb{R}^d \mid -\frac{L}{2} \leq x_i \leq \frac{L}{2} \text{ for all } i = 1, \ldots, d\} \tag{A.1.12}$$

$$= \{y \in \mathbb{R}^d \mid 0 \leq y_i \leq L \text{ for all } i = 1, \ldots, d\} \tag{A.1.13}$$

and extend the function $\psi : \Lambda_L \to \mathbb{R}$ to the set

$$R_L := \{y \in \mathbb{R}^d \mid -L \leq y_1 \leq L,\ 0 \leq y_i \leq L \text{ for all } i = 2, \ldots, d\} \tag{A.1.14}$$

by setting

$$\tilde{\psi}(y_1, y_\perp) := \begin{cases} \psi(y_1, y_\perp), & \text{for } y \in \Lambda_L, \\ 0, & \text{for } y \in R_L, y_1 = 0, \\ -\psi(-y_1, y_\perp), & \text{for } y \in R_L, y_1 < 0 \end{cases}$$

where $y_\perp = (y_2, \ldots, y_d)$. It is well known that the Laplacian of this extension is still in $L^2(R_L)$, an consequently $\tilde{\psi}$ is in the domain of the Dirichlet Laplacian on R_L, cf. e.g. [A] or [GT]. If we define $\check{V} : R_L \to \mathbb{R}$ by

$$\check{V}(y_1, y_\perp) := \begin{cases} V(y_1, y_\perp), & \text{for } y \in \Lambda_L, \\ 0, & \text{for } y \in R_L, y_1 = 0, \\ V(-y_1, y_\perp), & \text{for } y \in R_L, y_1 < 0 \end{cases}$$

then $\check{V}$ still takes values in $[-K, K]$ only, and the relation

$$H_L\tilde{\psi} = 0 \tag{A.1.15}$$

holds almost everywhere on R_L. Now we successively extend both $\check{V}$ and $\tilde{\psi}$ in the remaining $d-1$ directions and obtain functions defined on

$$\{y \in \mathbb{R}^d \mid -L \leq y_i \leq L \text{ for all } i = 1, \dots, d,\}$$

i.e. on a cube twice the side of the original one. Finally we extend these two functions periodically w.r.t. the lattice $(2L\mathbb{Z})^d$ to functions defined on $\mathbb{R}^d$ which satisfy (A.1.15) a.e. on $\mathbb{R}^d$. Moreover $\tilde{\psi}$ is in $W^{2,2}$ locally.

Note that if we restrict the extension $\tilde{\psi}$ to a cube of side $2kL$ with $k \in \mathbb{N}$ we obtain an L^2-eigenfunction of a box Schrödinger operator with Dirichlet b.c.

A.1.2 Dominating and weak boxes

Without loss of generality, we assume $K \in \mathbb{N}$ and $L \in K\mathbb{N}$. Up to boundaries (sets of measure zero) Λ_L can be decomposed into closed K-boxes

$$\Lambda = \bigcup_{k \in \tilde{\Lambda}} \Lambda_K(j)$$

where $\tilde{\Lambda} := K\mathbb{Z}^d \cap [-L/2, L/2)^d$.

Remark A.1.6. If the Delone parameter $K \notin \mathbb{N}$ or $L \notin K\mathbb{N}$, we take slightly larger boxes of side K' and L'. In the way we extend the eigenfunction ψ from Λ_L to the whole space $\mathbb{R}^d$, this does not affect our results.

Fix $T = 60K\lceil\sqrt{d}\rceil$, where $\lceil x \rceil$ stands for the least integer greater or equal than x. We say that the site $j \in \tilde{\Lambda}$ is *dominating*, if

$$\int_{\Lambda_K(j)} |\tilde{\psi}|^2 \geq \frac{1}{2(2T)^d} \int_{\Lambda_T(j)} |\tilde{\psi}|^2, \tag{A.1.16}$$

and call $\Lambda_K(j)$ a dominant box. Otherwise, we say the site j is *weak*, and call $\Lambda_K(j)$ a weak box. Notice that by the antisymmetry of $\tilde{\psi}$ we have

$$\int_{\Lambda_T(j)} \left|\tilde{\psi}\right|^2 \leq 2^d \int_{\Lambda_T(j) \cap \Lambda_L} \left|\tilde{\psi}\right|^2, \tag{A.1.17}$$

so that

$$\sum_{j \in \tilde{\Lambda}_L} \int_{\Lambda_T(j)} \left|\tilde{\psi}\right|^2 \leq (2T)^d \int_{\Lambda_L} \left|\tilde{\psi}\right|^2. \tag{A.1.18}$$

Then

$$\sum_{\text{weak sites}} \int_{\Lambda_K(j)} |\tilde{\psi}|^2 < \frac{1}{2(2T)^d} \sum_{\text{weak sites}} \int_{\Lambda_T(j)} |\tilde{\psi}|^2 \tag{A.1.19}$$

$$\leq \frac{1}{2(2T)^d} \sum_{j \in \Lambda_L} \int_{\Lambda_T(j)} |\tilde{\psi}|^2 \leq \frac{1}{2} \int_{\Lambda_L} |\tilde{\psi}|^2. \tag{A.1.20}$$

Thus the weak boxes contribute at most half of the total mass to the L^2 norm. Then,

$$2 \sum_{\text{dominant sites}} \int_{\Lambda_K(j)} |\tilde{\psi}|^2 > \int_{\Lambda_L(0)} |\tilde{\psi}|^2. \tag{A.1.21}$$

This means that it is sufficient to establish adequate unique continuation estimates for dominant boxes.

A.1.3 A unique continuation principle for dominant boxes and near-neighbor sites

Without loss of generality, assume that $L > 5K\lceil\sqrt{d}\rceil$. Fix a dominant box $\Lambda_K(j)$ and define the belt of near-neighbors of $\Lambda_K(j)$ as

$$A(j) = \Lambda_{2K\lceil\sqrt{d}\rceil+3K}(j) \setminus \Lambda_{2K\lceil\sqrt{d}\rceil+K}(j). \tag{A.1.22}$$

Note that under our assumption on L, $A(j) \neq \emptyset$. Indeed, this belt consists of $c_d := (2K\lceil\sqrt{d}\rceil+3K)^d - (2K\lceil\sqrt{d}\rceil + K)^d$ boxes of side K centered around points $j_n \in \tilde{\Lambda}_L$, where $n = 1, ..., c_d$. Note that for every $x \in A(j)$,

$$\text{dist}(x, \Lambda_K(j)) \geq K\sqrt{d} = \text{ diam } \Lambda_K(j). \tag{A.1.23}$$

We define the “right near-neighbor box” as the unique box $\Lambda_K(j_0) \subset A(j)$ such that

$$j_0 = j + (K\lceil\sqrt{d}\rceil + 2K)\,\mathbf{e}_1 \tag{A.1.24}$$

$$\Lambda_K(j_0) = \Lambda_K(j) + (K\lceil\sqrt{d}\rceil + 2K)\,\mathbf{e}_1 \tag{A.1.25}$$

where $\mathbf{e}_1 = (1, 0, 0, ...) \in \mathbb{R}^d$. That is, $\Lambda_K(j_0)$ is a translation of $\Lambda_K(j)$ in the positive direction along the first coordinate. By assumption, there exists a unique point of the Delone set $D = \{z_j\}_{j\in K\mathbb{Z}^d}$ contained in $\Lambda_K(j_0)$ that we denote by z_{j_0}. Moreover, $B(z_{j_0}, \delta) \subset \Lambda_K(j_0)$ and, since $z_{j_0} \in A(j)$, we have $\text{dist}(z_{j_0}, \Lambda_K(j)) \geq \text{diam}\,\Lambda_K(j)$ (see Fig A.1).

Note that both $B(z_{j_0}, \delta)$ and $\Lambda_K(j)$ are contained in the box $\Lambda_T(j)$. Next, we apply Corollary A.1.4 to $\tilde{\psi}$ in the dominant box and the ball $B(z_{j_0}, \delta)$ contained in its right near-neighbor.

Conditions (A.1.7) are satisfied taking

$$G = \Lambda_T(j), \quad \Theta = \Lambda_K(j) \tag{A.1.26}$$

and $B(z_{j_0}, \delta)$. Then, we have

$$R := \text{dist}(z_{j_0}, \Theta) \in [K\lceil\sqrt{d}\rceil, 2K\lceil\sqrt{d}\rceil]. \tag{A.1.27}$$

Here, we can take $\beta = 2(2T)^d$, since by the definition, for every dominant site j we have

$$\frac{\|\tilde{\psi}\|^2_{\Lambda_T(j)}}{\|\tilde{\psi}\|^2_{\Lambda_K(j)}} \leq 2(2T)^d =: \beta. \tag{A.1.28}$$

Then, by Corollary A.1.4, there exists a constant $C_{qUC} = C_{qUC}(d, K_{V_0,E}, R, \delta, \beta)$ such that for every dominant site j,

$$\|\tilde{\psi}\|^2_{B(z_{j_0},\delta)} \geq C_{qUC}\|\tilde{\psi}\|^2_{\Lambda_K(j)}. \tag{A.1.29}$$

Adding up all the dominant boxes, we obtain

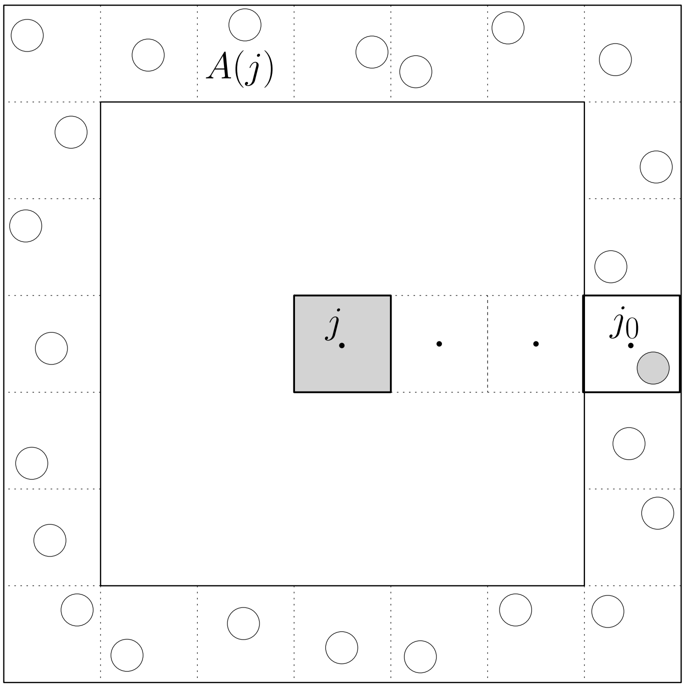

Figure A.1: The box $\Lambda_K(j)$ in $d=2$: its near-neighbor belt $A(j)$ contains the right near-neighbor with the unique Delone point z_{j_0}. The quantitative unique continuation principle from Corollary A.1.4 relates the mass of $\tilde{\psi}$ on the grey areas.

$$\sum_{\text{dominant sites}} \|\tilde{\psi}\|^2_{B(z_{j_0},\delta)} \geq C_{qUC} \sum_{\text{dominant sites}} \|\tilde{\psi}\|^2_{\Lambda_K(j)} \geq \frac{C_{qUC}}{2} \|\tilde{\psi}\|^2_{\tilde{\Lambda}_L}. \tag{A.1.30}$$

Note that in the l.h.s. of (A.1.30) there can be repeated terms $\|\psi\|^2_{B(z_{j_0},\delta)}$ because of the antisymmetric way of extending ψ to $\tilde{\psi}$ and the definition of "right near-neighbor". To see this, consider the extreme case where all boxes $\{\Lambda_K(j)\}_{k\in\tilde{\Lambda}_L}$ are dominant. For a box $\Lambda_K(j)$ that is in the boundary of Λ_L, its right near-neighbor box, say $\Lambda_K(j_0) \subset \mathbb{R}^d \setminus \Lambda_L$, is a mirror image of some box $\Lambda_K(j_0') \subset \Lambda_L$. This box, in turn, is the right near-neighbor of some other box in Λ_L (see Fig. A.2).

Then, the sum in the l.h.s. of (A.1.30) contains both terms $\|\tilde{\psi}\|^2_{B(z_{j_0},\delta)}$ and $\|\tilde{\psi}\|^2_{B(z_{j_0'},\delta)}$, where $z_{j_0} \in D \cap \Lambda_K(j_0) \subset D \cap \Lambda_L$, $z_{j_0'} \in D \cap \Lambda_K(j_0') \subset \mathbb{R}^d \setminus \Lambda_L$, and

$$\|\tilde{\psi}\|^2_{B(z_{j_0},\delta)} = \|\tilde{\psi}\|^2_{B(z_{j_0'},\delta)}. \tag{A.1.31}$$

Therefore, the sum in the l.h.s. of (A.1.30) contains at most 2 copies of each term $\|\tilde{\psi}\|^2_{B(z,\delta)}$, $z \in D \cap \Lambda_L$, that is,

$$\sum_{\text{dominant sites}} \|\tilde{\psi}\|^2_{B(z_{j_0},\delta)} \leq 2 \sum_{z\in D\cap\Lambda_L} \|\tilde{\psi}\|^2_{B(z,\delta)} = 2 \sum_{z\in D\cap\Lambda_L} \|\psi\|^2_{B(z,\delta)}. \tag{A.1.32}$$

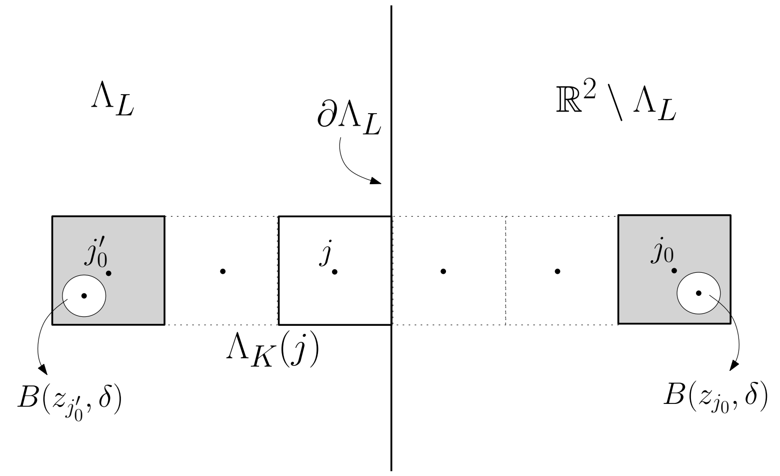

Figure A.2: A box near the boundary of Λ_L. In grey: its right near-neighbor and the corresponding mirror image in Λ_L.

This yields,

$$\sum_{z \in D \cap \Lambda_L} \|\psi\|^2_{B(z,\delta)} \geq \frac{C_{qUC}}{4} \|\psi\|^2_{\Lambda_L}. \tag{A.1.33}$$

The explicit dependence on K is given by Corollary A.1.4 and Remark A.1.5. This can also be seen directly in [GK11, Theorem A.1], where the dependence on R is written explicitly, which, for R as in (A.1.27), gives the desired result. Moreover, note that, by construction of $\tilde{A}(j)$, $R := \operatorname{dist}(z_j, \Lambda_K(j)) \in [K\lceil\sqrt{d}\rceil, K(\lceil\sqrt{d}\rceil)^2]$, then, from (A.1.10) we get that

$$C_{UCP} \geq C_{C,d,V_0,E} \frac{\delta^4}{K^2 \lceil\sqrt{d}\rceil^2} \left(\frac{\delta}{48K\lceil\sqrt{d}\rceil^2}\right)^{2\alpha} =: C_{UCP}(K), \tag{A.1.34}$$

where

$$\alpha = \max\{C_a, (C_b K^2_{V_0,E} \lceil\sqrt{d}\rceil^2)^{4/3} K^{4/3}, \ln(C_c(1 + K^2_{V_0,E}) \sqrt{2}(40\lceil\sqrt{d}\rceil)^{d/2} K^{d/2})\}. \tag{A.1.35}$$

Note that we can bound

$$\alpha \leq c_2 K^{4/3} + c_3 =: \alpha', \tag{A.1.36}$$

for some constants c_2, c_3 depending on the constants in A.1.35. Note moreover, that

$$\left(\frac{\delta}{48K\lceil\sqrt{d}\rceil^2}\right)^{2\alpha} \geq \left(\frac{\delta}{48K\lceil\sqrt{d}\rceil^2}\right)^{2\alpha'} \tag{A.1.37}$$

Then, from (A.1.34) we see that there exists a constant c_0 depending on constants coming from the Carleman estimate, d, δ and V_0, E such that

$$C_{UCP}(K) = c_1 K^{-c_2 K^{4/3} + c_3}. \tag{A.1.38}$$

□

A.2 Quantitative unique continuation principle for dimension 1

Let $I_L(x) = [x - L/2, x + L/2] \subset \mathbb{R}$ be the interval of length L centered in $x \in \mathbb{R}$, and consider the operator $H_0 = -\Delta + V_0$ restricted to $I_L(x)$ with Dirichlet boundary conditions, denoted by $H_{0,L}$. We recall a quantitative unique continuation principle obtained by [V96, KV02a] for eigenfunctions of $H_{0,L}$. The advantage of their proof is that it does not rely on the periodicity of the underlying configuration of supports, so it can be extended to a Delone-type configuration (for an extension to metric graphs, see [HV07, GHV08]). In order to know how the constant depends on the parameter R of a (r, R)-Delone set, in the spirit of Theorem A.1.1, we follow the proof of [KV02a, Lemma 5], writing explicitly the constants involved. In this way, we obtain an explicit lower bound for the concentration of an eigenfunction on small balls supported in points of a (r, R)-Delone set.

Theorem A.2.1. *Let $I \subset \mathbb{R}$ be a bounded interval and let ψ be an eigenfunction of the operator $H_{0,L} = -\Delta_L + V_{0,L}$ with eigenvalue $E \in I$ on $L^2(I_L(x))$, that is, $-\psi'' + V_{0,L}\psi = E\psi$. Fix $s > 0$, and decompose the interval $I_L(x)$ into $\frac{L}{R}$ intervals of length R, $I_R(k)$ for $k \in \mathcal{J}_{x,L,R}$ for some set $\mathcal{J}_{x,L,R}$. There exists a constant C^1_{UCP} such that*

$$\int_{\Lambda_s(\gamma_k)} |\psi|^2 \geq C^1_{UCP} \int_{\Lambda_R(k)} |\psi|^2 \tag{A.2.1}$$

where the constant C^1_{UCP} is uniform on $L \in \mathbb{N}$ and γ_k is an arbitrary point in $I_R(k)$. Moreover, if we define

$$V(x) = \sum_{k \in \mathcal{J}_{x,L,R}} u(x - \gamma_k), \tag{A.2.2}$$

for a normalized eigenfunction of $H_{0,L}$, where $u \geq u_- \chi_{I_s(0)}$. we have

$$\langle V\psi, \psi \rangle \geq u_- \cdot C^1_{UCP} \tag{A.2.3}$$

The explicit form of the constant is

$$C^1_{UCP} = \frac{s}{R} e^{-2C_{s,V_0,E}R}. \tag{A.2.4}$$

Before proving this Lemma, we restate an intermediate result from [KV02a, Lemma 5], writing explicitly the constants involved.

Lemma A.2.2. *Fix $s > 0$, $s < R < L$. For ψ the eigenfunction above, and any $k \in I_L(x)$ such that $I_R(k) \subset I_L(x)$, there exists a constant $C_{s,V,E}$ such that*

$$\|\psi\|^2_{I_s(k+y)} \leq e^{C_{s,V,E}|y|} \|\psi\|^2_{I_s(k)} \tag{A.2.5}$$

for any $y \in I_{R-s}(0)$, where

$$C_{s,V_0,E} := \frac{4 \cdot 18^2}{s^2} + 4s^2 \|V_0 - E\|_\infty. \tag{A.2.6}$$

Proof. We follow the same proof as in [KV02a]. Define, for $k \in I_L(x)$

$$\phi(y) := \int_{I_s(k+y)} |\psi|^2 \text{, for } y \in I_R(0), \tag{A.2.7}$$

then

$$|\partial_y \phi(y)| \leq 2\|\psi\|_{I_s(k+y)}\|\psi'\|_{I_s(k+y)}. \tag{A.2.8}$$

By standard Sobolev estimates (see the proof of [GT, Theorem 7.25])

$$\|\psi'\|_{I_s(k+y)} \leq \frac{2\cdot 18^2}{s^2}\|\psi''\| + 2s^2\|\psi\|_{I_s(k+y)}. \tag{A.2.9}$$

By the eigenfunction equation,

$$\|\psi'\|_{I_s(k+y)} \leq \left(\frac{2\cdot 18^2}{s^2} + 2s^2\|V_0 - E\|_\infty\right)\|\psi\|_{I_s(k+y)}, \tag{A.2.10}$$

so

$$|\partial_y \phi(y)| \leq \left(\frac{4\cdot 18^2}{s^2} + 4s^2\|V_0 - E\|_\infty\right)\|\psi\|_{I_s(k+y)}. \tag{A.2.11}$$

That is,

$$|\partial_y \phi(y)| \leq C_{s,V,E}\phi(y), \tag{A.2.12}$$

where

$$C_{s,V_0,E} := \frac{4\cdot 18^2}{s^2} + 4s^2\|V_0 - E\|_\infty. \tag{A.2.13}$$

Then, by Gronwall's Lemma, $|\partial_y \phi(y)| \leq e^{C_{s,V,E}|y|}\phi(0)$, that is,

$$\|\psi\|^2_{I_s(k+y)} \leq e^{C_{s,V_0,E}|y|}\|\psi\|^2_{I_s(k)}. \tag{A.2.14}$$

□

Proof of Theorem A.2.1. Now, decompose $I_L(x)$ into $\frac{L}{R}$ intervals of side R, $I_R(k)$ for $k \in \mathcal{J}_{x,L,R}$. Furthermore, decompose each interval $I_R(k)$ into $\frac{R}{s}$ smaller intervals of length s, $I_s(k+y_j)$ with $y_j \in \mathcal{J}_{k,R,s}$. Notice that $|y_j| \leq R$ for all j.

By Lemma A.2.2 applied to the centers $k \in \mathcal{J}_{x,L,R}$ we have

$$\begin{aligned}\|\psi\|^2_{I_R(k)} &= \sum_{y_j\in\mathcal{J}_{k,R,s}} \|\psi\|^2_{I_s(k+y_j)} \\ &= \sum_{y_j\in\mathcal{J}_{k,R,s}} e^{C_{s,V,E}|y_j|}\|\psi\|^2_{I_s(k)} \\ &\leq \frac{R}{s}\, e^{C_{s,V,E}R}\,\|\psi\|^2_{I_s(k)},\end{aligned} \tag{A.2.15}$$

where we used the fact that $|\mathcal{J}_{k,R,s}| \leq \frac{R}{s}$.

Now, since Lemma A.2.2 is true for intervals centered in any point of $I_L(x)$ and $y \in I_{R-s}(0)$, in particular for an arbitrary point $\gamma_k \in I_{R-s}(k)$,

$$\begin{aligned}\|\psi\|^2_{I_s(k)} = \|\psi\|^2_{I_s(\gamma_k-(\gamma_k-k))} &\leq e^{C_{s,V,E}|\gamma_k-k|}\,\|\psi\|^2_{I_s(\gamma_k)} \\ &\leq e^{C_{s,V,E}R}\,\|\psi\|^2_{I_s(\gamma_k)},\end{aligned} \tag{A.2.16}$$

where we used that $|\gamma_k - k| \leq R$. Inserting this in A.2.15 gives

$$\|\psi\|^2_{I_R(k)} \leq \frac{R}{s}\, e^{2C_{s,V_0,E}R}\, \|\psi\|^2_{I_s(\gamma_k)}. \tag{A.2.17}$$

Therefore

$$\|\psi\|^2_{I_L(x)} = \sum_{k\in\mathcal{J}_{x,L,R}} \|\psi\|^2_{I_R(k)} \leq \frac{R}{s} e^{2C_{s,V_0,E}R} \sum_{k\in\mathcal{J}_{x,L,R}} \|\psi\|^2_{I_s(\gamma_k)}. \tag{A.2.18}$$

For a potential V defined by (A.2.2), this implies

$$\langle V\psi, \psi\rangle \geq u_- \sum_{\gamma_k\in\mathcal{J}_{x,L,R}} \|\psi\|^2_{I_s(\gamma_k)} \geq u_-\, C^1_{UCP}\, \|\psi\|^2_{I_L(x)}, \tag{A.2.19}$$

where C^1_{UCP} denotes

$$C^1_{UCP} = \frac{s}{R}\, e^{-2C_{s,V,E}R}. \tag{A.2.20}$$

□

A.3 Perturbation of the ground state energy for finite-volume operators and its consequences

We restate [RMV12, Theorem 4.9-(a)], for a Delone potential with an underlying (r, R)-Delone set.

Theorem A.3.1. *Let $V_0\colon \mathbb{R}^d \to \mathbb{R}$ be bounded and measurable, $t \in (0, 1]$, $\delta \in (0, R/4]$, $\{z_k\}_{k\in R\mathbb{Z}^d} \subset \mathbb{R}^d$ a sequence such that*

$$\forall\ k \in R\mathbb{Z}^d : \quad B(z_k, \delta) \subset \Lambda_R(k)$$

Let χ denote the characteristic function of $\bigcup_{k\in R\mathbb{Z}^d} B(z_k, \delta)$ and $W \geq C_- \cdot \chi$ a bounded potential with $C_- > 0$. For $L \in \mathbb{N}$ and $x \in \mathbb{R}^d$, denote by $\lambda_{x,L}(t) = \inf \sigma(H_{t,x,L})$ the bottom of the spectrum of $H_{t,x,L} := -\Delta + V_0 + tW$ on $\Lambda_L(x)$ with periodic or Dirichlet boundary conditions. Then

$$\forall\ t \in (0, 1] : \quad \lambda_{x,L}(t) \geq \lambda_{x,L}(0) + \kappa \cdot t$$

where $\kappa := C_-{\cdot}C_{sfUC}$ and C_{sfUC} is the constant from the scale-free unique continuation principle.

Here, C_{sfUC} is given in Theorem A.1.1.

Remark A.3.2.

i. In our applications, W is the Delone potential associated to a (r, R)-Delone set D, $D = \{z_k\}_{k\in R\mathbb{Z}^d}$. Therefore, the previous result states that a Delone-type perturbation of an operator $H_{0,L} = -\Delta_L + V_{0,L}$, where V_0 is bounded, produces a gap in the spectral infimum of size proportional to constant in the quantitative unique continuation principle from Theorem A.1.1.

ii. In dimension $d = 1$ we can use Theorem A.2.1 to obtain the same result, but with $C_{sfUC,d}$ replaced by the constant C^1_{UCP} defined in (A.2.4).

Proof. The normalized ground state $\psi(t)$ of $H_{t,L,x}$ satisfies

$$\lambda^{L,x}(t) = \inf \sigma(H_{t,L,x}) = \langle \psi(t), H_{t,L,x}\psi(t)\rangle \tag{A.3.1}$$

Note that the family of potentials $V_0 + tW$, $t \in [0,1]$ is uniformly bounded. Thus the set $\{\lambda^{L,x}(t) : t \in [0,1], L \in \mathbb{N}\}$ is contained in a compact interval $\mathfrak{I} \subset \mathbb{R}$. We will apply the scale-free unique continuation principle to this family of potentials. Note that the parameter K which enters the constant C_{sfUC} is bounded uniformly by the finite number $\sup\{\|V_0 + tW - E\|_\infty;\ E \in \mathfrak{I},\ t \in [0,1]\}$. It follows

$$\lambda^{L,x}(t) = \langle \psi(t), H_{t,L,x}\psi(t)\rangle = \langle \psi(t), H_{0,L,x}\psi(t)\rangle + t\langle \psi(t), W\psi(t)\rangle \tag{A.3.2}$$
$$\geq \langle \psi(t), H_{0,L,x}\psi(t)\rangle + C_- \cdot C_{sfUC}\, t \tag{A.3.3}$$
$$\geq \lambda^{L,x}(0) + t\kappa \tag{A.3.4}$$

where in the last line we used the min-max principle on the operator $H_{0,L,x}$. In particular, for $t = 1$,

$$\lambda^{L,x}(1) \geq \lambda^{L,x}(0) + \kappa. \tag{A.3.5}$$

□

Recall that we denote by $P_{0,L}(I)$ the spectral projector of $H_{0,L}$ associated to the interval I. Next, we recall [RMV12, Corollary 4.10]:

Corollary A.3.3. *Let the hypotheses and notation of Theorem A.3.1 hold and assume that Dirichlet boundary conditions are imposed. In this case, we know that for any $x \in \mathbb{R}^d$ and $L \in \mathbb{N}$*

$$\lambda^{L,x}(0) \geq E_{\min} := \inf \sigma(-\Delta + V_0).$$

Then

$$\forall\ t \in (0,1] : \quad \lambda^{L,x}(t) \geq E_{\min} + \kappa \cdot t$$

Fix $q \in (0,1)$ and set $I = (-\infty, E_{\min} + q\kappa]$. Then the following uncertainty principle holds

$$P_{0,L}(I)WP_{0,L}(I)geq(1-q)\kappa P_{0,L}(I)$$

By taking $q = 1/2$, we have that

$$P_{0,L}(I)VP_{0,L} \geq \frac{\kappa}{2}P_{0,L}. \tag{A.3.6}$$

This estimate is relevant for the proof of an optimal Wegner estimate, where the Wegner constant $Q_W = (\kappa/2)^{-2}Q$, for some constant Q depending on the parameters of the model, as can be seen in (3.2.24) (see e.g. [CHK07]).

A.4 The Brownian bridge

In order to make Chapter 5 self-contained, we recall of the definition of Brownian bridge and the Feynman–Kac–Îto formula used in the proof of Lemma 5.2.2. We recall results from [BrLM04, Section 1.2].

Let $t > 0$ be fixed. The Brownian bridge $\mathbf{b}(s)$ associated to the pair $(x, y) \in \mathbb{R}^d \times \mathbb{R}^d$ and the time interval $[0, t]$ is a stochastic process with values in $\mathbb{R}^d$ starting in $x \in \mathbb{R}^d$ and ending in $y \in \mathbb{R}^d$. That is, the vector $\mathbf{b}(s) = (b_1(s), b_2(s), ..., b_d(s))$ is such that $\mathbf{b}(0) = x$ and $\mathbf{b}(t) = y$. Its components $b_j(s)$ are independent, continuous functions of $s \in [0, t]$ and are distributed according to a Gaussian probability measure with mean

$$s \mapsto x_j + (y_j - x_j)\frac{s}{t}, \quad \text{for } s \in [0, t], \tag{A.4.1}$$

and covariance

$$(s, u) \mapsto \min\{s, u\} - \frac{s\,u}{t} \quad \text{for } (s, u) \in [0, t] \times [0, t]. \tag{A.4.2}$$

The joint probability measure is denoted by $\mu^{0,t}_{x,y}$.

Consider the operator $H = (-i\nabla - \mathbf{A})^2 + V$. Under some regularity conditions on $\mathbf{A}$ and V (see [BrLM04, Definition 1.1]), we can define the Euclidian action functional

$$S_t(A, V; b) = i\int_0^t \mathrm{d}\mathbf{b}(s) \cdot \mathbf{A}(\mathbf{b}(s)) + \frac{i}{2}\int_0^t (\nabla \cdot \mathbf{A}(\mathbf{b}(s))\mathrm{d}s + \int_0^t V(\mathbf{b}(s))\mathrm{d}s). \tag{A.4.3}$$

In [BrLM04, Theorem 1.10] it was proved that e^{-tH} is a well defined operator, given by the Feynman–Kac–Îto formula

$$e^{-tH}\varphi = \int_{\mathbb{R}^d} k_t(\cdot, y)\varphi(y)\mathrm{d}y, \tag{A.4.4}$$

where the continuous integral kernel $k_t(x, y)$ is given by

$$k^H_t(x, y) := \frac{e^{-|x-y|^2/2t}}{(2\pi t)^{d/2}} \int \mu^{0,t}_{x,y}(db)e^{-S_t(A, V_{P\omega}; b)}. \tag{A.4.5}$$

The properties of $k_t(x, y)$ are listed in [BrLM04, Lemma 1.7]. In particular, we note that $k_t(x, y)$ is Hermitian, that is, $k_t(x, y) = k^*_t(y, x)$ for all $x, y \in \mathbb{R}^d$ and for every $x \in \mathbb{R}^d$, $k_t(x, \cdot) \in \mathrm{L}^2(\mathbb{R}^d)$.

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